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Purity And Projectivity Relative To A Submodule

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Abstract

In the literature there are some links between the notions of directsummands, purity and projectivity. In this paper, we study these links among the notions but relative to an arbitrary submodules, also a partial links are consider. Properties and characterizations of these notions which are correspondence, have been given.Among others, the following have been proved. If M is a projective module, every pure submoduleN relative to a submodule T is projective relative to N ∩ T. Also if Mis an R-module with R is M-injective, then every projective submoduleof M relative to T is pure relative to T. As a consequence of the above. If M ⊕ R is quasi-continuous, then M is injective and every projectivesubmodule relative to T is pure relative to T.

Keywords: (T)-direct sumands, (T)-pure submodules, (T)-projective modules

1- INTRODUCTION

In what follows R will denote an associative ring with non-zero identity andan R-module will mean unitary left R-module. Recall that an R-module Pis projective, if give any R-epimorphism$\alpha : A \to B$, any R-homomorphism $\beta : P \to B$ can be lifted to an R-homomorphism $h : P \to A$ along β, thatis $\alpha \circ h(x) = \beta(x)$ for all x in P. As a consequence of the vital role thatprojectivity occupies in various parts of mathematics,

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several generalizations ofprojectivity (e.g M-projectivity, quasi-projectivity, im-projectivity, etc[7],[8],[2])have appeared which center around completing a diagram of R-modulesand R-homomorphisms is an elementwise or in a submodules sense. Other versions of generalizations of projectivity have recently appeared which center around the existence of an R-homomorphism h (above) along R-epimorphism (or R-epimorphism with small kernal) β [6] The purpose of this paper is to initiate the study of projectivity relative to a submodule, in other words we study R-modules which are define in such away that a projective-type diagram is completing in an elementwise sense but relative to a submodule. Let M be an R-module and T a submodule of M. M is called projective relative to T, if given R-epimorphism$f : A \to B$, for any R-homomorphism $g : M \to B$, there exists an R-homomorphism $h : M \to A$ such that $f \circ h(x)g(x) \in g(T)$ for all x in M. Characterization of projective modules relative to a submodule in terms of their presentation is given . We investigatetheir dual basis. The concept of purity (in the sense of cohn [3])have been related with projectivity. Thus we introduce pure submodule (as well as direct summands) relative to a submodule and have been related with projectivity relative to a submodule. Finally, a criteria for pure submodules relative to a submodule have been suggested.

2- PURE SUBMODULE RELATIVE TO A SUBMODULE

The notion of purity for abelian groups are generalized to modules over arbitrary rings in several ways, of which the best-known is Cohn's purity [3]. In this section we introduce purity relative to a submodule as a generalization of Cohn's purity.

Definition 2.1 Let M be an R-module and T a submodule of M. A submodule N of M is said to be pure relative to T (Simply (T)-pure) if for each ideal A of R, $AM \cap N = AN + T \cap (AM \cap N)$

For any submodule N of an R-module M, N is (N)-pure. It is clear tosee the following; a submodule N is pure if and only if N is (0)-pure. If N is (T_1)-pure in M, then N is (T_2)-pure for every submodule T_2 of M containing T_1, thus every pure submodule of M is (T)-pure for every submodule T of M. The converse may not be true in general, the submodule$\bar{2}Z_4$ in the Z_4-module$Z_4$is ($\bar{2}Z_4$)-pure but not pure. However, if N is (T)-pure in M and $N \cap T = 0$, then N is pure.

The following gives an equational characterization of (T)-pure submoduleswhich is more usable than the definition.

Proposition 2.2: let M be an R-module and T a submodule of M. Then asubmodule N of M is (T)-pure if and only if for every finite sets $\{m_i\} \subseteq M$,$\{n_i\} \subseteq N$ and $\{r_{ij}\} \subseteq R$ with $nj = \sum_{i=1}^{t} r_{ij}\, m_i$, j = 1, 2, . . . , k, there is a finite set $\{x_i\} \subseteq N$ such that$n_j - \sum_{i=1}^{t} r_{ij}\, x_i \in T \cap N$

Proof: Let A be the left ideal generated by r_{ij} ($1 \leq i \leq t$ and $1 \leq j \leq k$). Then $nj \in AM \cap N$. There exist $x_i \in N$ and $w_j \in T \cap N$ such that

$n_j = \sum_{i=1}^{t} r_{ij}\, x_i + w_j$, so $n_j - \sum_{i=1}^{t} r_{ij}\, x_i \in T \cap N$. Conversely, let A be a left ideal of R and $b \in AM \cap N$. Then $b = \sum_{i=1}^{t} a_i\, m_i$ where $a_i \in A$ and $m_i \in M$. There exists a finite set $\{x_i\} \subseteq N$ such that $b - \sum_{i=1}^{t} a_i\, x_i \in T \cap N$,but $b - \sum_{i=1}^{t} a_i\, x_i \in AM$, hence $b \in AN + T \cap (AM \cap N)$.

By looking carefully at the proof of the above proposition, we see that, if M is an R-module and T a submodule of M, then a submodule N of Mis (T)-pure if and only if $AM \cap N = AN + T \cap (AM \cap N)$ for each finitely generated left ideal A of R.

As an application of the above proposition, we can easily verifying the following. Let M be an R-module and T a submodule of M. Then

1. (T)-pure submodule of a pure submodule of M is (T)-pure in M.
2. If N is (T)-pure in M, then N is (T ∩ W)-pure in every submoduleW of M containing N.
3. If N_i is (T_i)-pure in M_i (i = 1, 2, . . . , n), then $\oplus_i N_i$ is $(\oplus_i T_i)$-pure in $\oplus_i M_i$.
4. Union of ascending chain of (T)-pure submodules of M is (T)-pure.

It is well-known that, every direct summand of an R-module M is pure,hence every direct summand is (T)-pure for every submodule T of M. Weintroduce a relative direct summand to a submodule of M

Definition 2.3: Let T be a submodule of an R-module M. Then a submoduleN of M is (T)-direct summand, if there exists a submodule K of M with M = N + K and N ∩ K ⊆ T.

Clearly, N is a direct summand of M if and only if N is (0)–summandand hence every direct summand in M is (T)-summand for every submoduleT of M, but in the Z-module Z_{12}, $\bar{2}Z_{12}$ is $(\bar{6}Z_{12})$-direct summand and it is not direct summand.

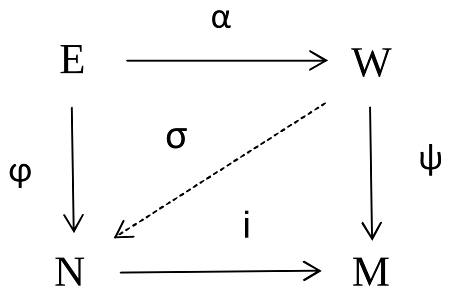
Proposition 2.4: Let T be a submodule of an R-module M. Then every (T)-summand of M is (T)-pure.

Proof: Let P be a (T)-summand of M. Then there exists a submoduleQ of M with M = P + Q and P ∩ Q ⊆ T. Let $x_j = \sum r_{ij} m_i \in P$ where $m_i \in M$, $r_{ij} \in R$, $1 \le i \le t$ and $1 \le j \le k$. For each i, $m_i = p_i + q_i$ where $p_i \in P$ and $q_i \in Q$,$x_j - \sum r_{ij} p_i = \sum r_{ij} q_i \in P \cap Q \subseteq T$.

Let R = ⊕Zp where the direct sum runs over all primes. It is clear that Ris Von Neumman regular ring. The ideal $A = \oplus_{p>2} Z_p$ is not finitely generated.Then A is (B)-pure ideal of R for each ideal B of R, but A is not (B)-directsummand, otherwise A is finitely generated [9].

Another characterization of (T)-pure submodules is in the following.

Proposition 2.5: Let M be an R-module and T a submodule of M. Then asubmodule N is (T)-pure in M if and only if for every commutative diagram ofR-modules and R-homomorphisms



Where E and W are free R-modules and E is finitely generated there is an R-homomorphism σ : W → N such that $\sigma \circ \alpha$ (e) − φ(e) ∈T for each e in E.

Proof: Since α(E) is contained in some finitely generated free R-modulewhich is a direct summand of W, we can assume without loss of generalitythat W is finitely generated free module. Thus, if $\{e_j\}$ and $\{w_i\}$ are bases forE and W respectively, j = 1, 2, . . . , n , i = 1, 2, . . . ,m and $\alpha(e_j) = \sum a_{ij} w_i$,then the commutativity of the diagram means that we have a finite system ofequations,$n_j = \sum a_{ij} m_i \in N$ with $m_i = \psi(w_i)$ and $n_j = \varphi(e_j)$. Since N is(T)-pure there are $\acute{n}_i \in N$ such that $n_j - \sum a_{ij} \acute{n}_i \in T \cap N$. Put $\sigma(w_i) = \acute{n}_i$ and extend σ to all W. It is clear that $\sigma \circ \alpha$ (e) − φ(e) ∈ T for all e ∈ E.

Conversely, if $n_j = \sum a_{ij} m_i$ is a finite system of equations solvable in M, thenit is easily to construct a commutative diagram of the above type. If $\{w_i\}$ is a basis for W, then σ(wi) is a solution in N relative to T.

3- PROJECTIVE MODULES RELATIVETO A SUBMODULE

In this section, we consider some weak form of projectivity and we try to studythe most important results concerning these modules.
Let M be an R-module and T a submodule of M. Then M is said to be projective relative to T (simply, (T)-projective), if for each R-epimorphism $f : A \rightarrow B$ and R-homomorphism $g : M \rightarrow B$, there exists an R-homomorphism $h : M \rightarrow A$ such that $f \circ h(m) - g(m) \in g(T)$ for each m in M. Clearly, an R-module M is projective if and only if M is (0)-projective. If M is a (T_1)-projective R-module, then M is (T_2)-projective for each submoduleT_2 of M containing T_1. Thus every projective R-module is (T)-projective for each submodule T of M. Every R-module M is (M)-projective, thus (T)-projective R-module may not be projective. Also it is easy to check by the definition that, if M1 is a (T)-projective R-module and $\alpha: M_1 \rightarrow M_2$is an isomorphism, then M_2 is $(\alpha(T))$-projective.

Remark 3.1:

(a) If M is a (T)-projective R-module and N a direct summand of M then N is $(\rho(T))$-projective where ρ is the projection of M onto N.

Proof: Let $M = N \oplus W$. For each R-epimorphism $\alpha : A \rightarrow B$ andR-homomorphism $\beta : N \rightarrow B$, there exists an R-homomorphism $k : M \rightarrow A$ such that $\alpha \circ k(m) - \beta \circ \rho(m) \in \beta \circ \rho(T)$ for each $m \in M$ where $\rho : M \rightarrow N$ is the natural projection of M onto N. Now, for each $n \in N$, $j(n) \in M$ where j is the injection of N into M with $\rho \circ j = I_N$. Thus $\alpha \circ k \circ j(n) - \beta(n) \in \beta(\rho(T))$. This shows that N is $(\rho(T))$-projective.

(b) The following two results follow from the above remark

(i) If M is a (T)-projective R-module and N is a direct summand of M with $N \cap T = 0$, then N is projective.

(ii) If M is a (T)-projective R-module and T a direct summand of M, then every intersection complement of T is projective.

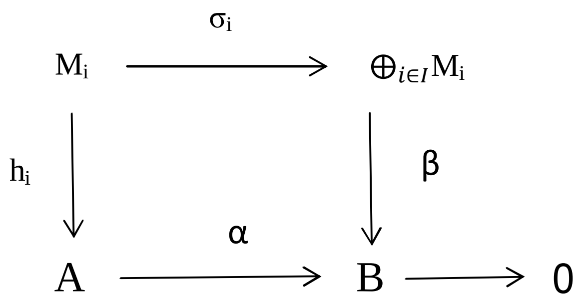
(c) Recall that asubmodule N of M is stable if $\alpha(N) \subseteq N$ for each R-homomorphism

$\alpha : N \rightarrow M$[1]. If M is a (T)-projective R-module and T is an epimorphic image of a stable submodule T_1 of M, then M is (T_1)-projective.

Proposition 3.2: Let $\{m_i\}_{i\in I}$ be an arbitrary family of R-modules M_i andT_i a submodule of M_i for each $i \in I$. Then $\oplus_{i\in I}M_i$ is $(\oplus_{i\in I}T_i)$-projective if and only if M_i is (T_i)-projective for each $i \in I$.

Proof: Let $\alpha : A \rightarrow B$ be an R-epimorphism and $\beta : \oplus_{i\in I}M_i \rightarrow B$ be an R-homomorphism. For each $i \in I$, define $\sigma_i : M_i \rightarrow \oplus_{i\in I}M_i$ by $\sigma_i(m)(j) = \begin{cases} m & \text{if } i=j \\ o & \text{if } i \neq j \end{cases}$

Since M_i is (T_i)-projective, then there is an R-homomorphism $h_i : M_i \rightarrow A$ such that $\alpha \circ h_i(m) - \beta \circ \sigma_i(m) \in \beta \circ \sigma_i(T_i)$, for each $i \in I$ and $m \in M_i$



Let $f \in \oplus_{i\in I}M_i$. Then $f : I \rightarrow \cup_{i\in I} M_i$ such that $f(i) \neq 0$ only for finitely many $i \in I$. Define $k : \oplus_{i\in I}M_i \rightarrow A$ by $k(f) = \sum_{i\in I} h_i f(i)$.It is clear that k is a well - defined R-homomorphism. Observe that $(\sigma_i \circ f(i))(i) = f(i)$ for each $i \in I$.$\alpha \circ k(f) - \beta(f) = \alpha(\sum_{i\in I} h_i f(i)) - \beta(\sum_{i\in I} h_i f(i)) = \sum_{i\in I} \alpha \circ h_i f(i) - \beta(\sum_{i\in I} \sigma_i(f(i))) = \sum_{i\in I}(\alpha \circ h_i f(i) - \beta \circ \sigma_i f(i)) \in \sum_{i\in I} \beta \circ \sigma_i(T_i) = \beta(\oplus_{i\in I} T_i)$

This shows that $\oplus_{i\in I}M_i$ is $(\oplus_{i\in I}T_i)$-projective. The other direction follows from Remark (3.1)(a)

In the following we give a criteria of (T)-projective modules in terms oftheir presentation.

Theorem 3.3: Let M be an R-module and T a submodule of M. Then M is(T)-projective if and only if for each short exact sequence

$$0 \longrightarrow K \xrightarrow{i} W \xrightarrow{\pi} M \longrightarrow 0$$

Where W is a free R-module and K = ker(π), there is an R-homomorphism θ : W →W such that

1. K ⊆ ker(θ) and

2. π ∘ θ(x) − π(x) ∈ T ∀ x ∈ W

Proof: (T)-projectivity of M implies that there is an R-homomorphism : W → M such that π ∘ h(m) − m ∈ T for each m ∈ M. Then θ = h∘ π : W → W is an R-homomorphism and for each w ∈ W, π ∘ θ(w) − π(w) = π ∘ h(π(w)) − π(w) ∈T and it is clear that K ⊆ ker(θ). Conversely, consider an R-epimorphism f : A → B and R-homomorphism g : M → B. Let $\{m_\alpha \mid \alpha \in \Lambda\}$be a generated set of M and W be a free R-module with basis $\{x_\alpha \mid \alpha \in \Lambda\}$. Define π : W → M by π(w) =$\sum_{i=1}^{n} r_i m_{\alpha_i}$ for each w = $\sum_{i=1}^{n} r_i x_{\alpha_i}$ in W. In particular

$$\pi(x_\alpha) = m_\alpha \forall\ \alpha \in \Lambda (1)$$

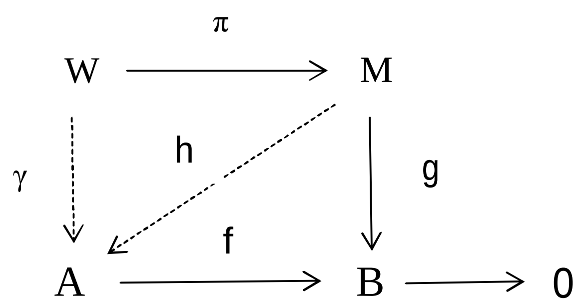
By hypothesis, there is an R-homomorphism θ : W → W such that ker(π) ⊆ ker(θ) and π ∘ θ (x) − π(x) ∈ T for each x ∈ W. Define ψ : M →W by $\psi(m) = \psi\ (\sum_{j=1}^{k} s_j m_{\alpha_j}) = \sum_{j=1}^{k} s_j\, \theta(x_{\alpha_j}) = \theta(\sum_{j=1}^{k} s_j\, x_{\alpha_j})$ where m ∈ M.Inparticular

$$\psi(m_\alpha) = \theta\ (x_\alpha) \qquad \forall\ \alpha \in \Lambda\ (2)$$

If$\sum_{j=1}^{k} s_j m_{\alpha_j} = \sum_{t=1}^{n} r_t m_{\beta_t}$, then$\sum_{j=1}^{k} s_j\, \pi(x_{\alpha_j}) = \sum_{t=1}^{n} r_t\, \pi(x_{\beta_t})$

$\sum_{j=1}^{k} s_j\ (x_{\alpha_j}) - \sum_{t=1}^{n} r_t\ (x_{\beta_t})$ ∈ ker(π)⊆ ker(θ), hence ψ is well defined.

Consider the following diagram



By freeness of W, there is an R-homomorphism $\gamma : W \to A$ such that $f \circ\gamma= g \circ \pi$. Define $h : M \to A$ by putting $h = \gamma \circ\psi$. Now, for each $m_{\alpha_0} \in\{m_\alpha \mid \alpha \in\Lambda\}$and using (1) and (2) we have the following $f \circ h(m_{\alpha_0}$ $- g(m_{\alpha_0}) = f \circ\gamma \circ \psi(m_{\alpha_0}) - g(m_{\alpha_0}) = g \circ \pi(\theta(x_{\alpha_0}) - g(\pi(x_{\alpha_0})) = g(\pi \circ \theta(x_{\alpha_0}) - \pi(x_{\alpha_0}))\in g(T)$. Thus M is (T)-projective.

Recall that an R-module X is im-projective if given any R-epimorphism $f : A \to B$ and any R-homomorphism $g : X \to B$, there exists an R-homomorphism $h : X \to A$ such that $g(X) \subseteq f \circ h(X)$ [2]. A submoduleN of an R-module M is called small, if $N + K \neq M$ for each proper submoduleK of M [7].

In the following we see that the class of (T)-projective modules is containedin that of im-projective modules for certain class of submodules.

Corollary 3.4: Every projective module relative to a small submodule is an im-projective.

Proof: Let M be a (T)-projective R-module where T is a small submodule of M consider a short exact sequence

$$0 \longrightarrow K \xrightarrow{\ i\ } W \xrightarrow{\ \pi\ } M \longrightarrow 0$$

Where W is a free R-module and $K = \ker(\pi)$. By Theorem (3.3), there is an R-endomorphism θ of W such that $\pi \circ\theta(x)-\pi(x) \in T$ for each x in W. Hence $M = \pi(W) = \pi\circ\theta(W)+T$. Smallest of T implies that $M = \pi \circ \theta(W) = \pi(W)$. By [2], M is im-projective.

An R-module M is hopfian if every onto endomorphism of M is automorphism.It is proved in [2], That hopfianim-projective R-module is projective, thus we have.

Corollary 3.5: Let M be an R-module and T a small submodule of M. If M is hopfian (T)-projective, then M is projective.

As is well-known, every finitely generated R-module is hopfian.

Corollary 3.6 Every finitely generated projective module relative to a small submodule is projective.

Definition 3.7: Let A and B be R-modules and T a submodule of B. An R-homomorphism $\alpha : A \to B$ is called (T)-split, if there exists an R-homomorphism $\beta : B \to A$ such that $\alpha \circ \beta(b) - b \in T$ for each $b \in B$.

In the following, we consider another characterization of (T)-projective modules, statement (3) is so called the dual basis lemma of a (T)-projective modules

Theorem 3.8: The following statements are equivalent for an R-module M and a submodule T of M

1. M is (T)-projective,
2. Every R-epimorphism $\alpha : A \to M$ is (T)-split for each R-module A,
3. For every family $\{m_i \mid i \in I\}$ of generators of M, there exists a family $\{f_i \mid i \in I\} \subseteq M^* = Hom_R(M,R)$ such that for each $m \in M$

(a) $f_i(m) \neq 0$ for only finitely many $i \in I$

(b) $\sum_{i\in I} f_i(m)m_i - m \in T$.

Proof: (1) $\Longrightarrow$ (2) Let A be an arbitrary R-module and $\alpha : A \to M$ be anR-epimorphism. By (T)-projectivity of M, there exists an R-homomorphism $\beta : M \to A$ such that $\alpha \circ \beta(m) - m \in T$ for each $m \in M$.

(2) $\Longrightarrow$ (3) Let $\{m_i \mid i \in I\}$ be a generated set of M and let W be a free R-module with $\{w_i \mid i \in I\}$. Define $\psi : W \to M$ by $\psi(y) = \psi\left(\sum_{i=1}^{n} r_i w_i\right) = \sum_{i=1}^{n} r_i m_i$ for each $y \in W$. In particular $\psi(w_i) = m_i$ for each $i \in I$. It is clear that ψ is well defined R-epimorphism. Hence by (2), there is an R-homomorphism $g : M \to W$ such that $\psi \circ g(m) - m \in T$ for each $m \in M$.

Now for each $i \in I$, define $\delta_i : W \to R$ by $\delta_i(w_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

It is easy to check that δ_i is an R-homomorphism and for each $y \in W$, $y = \sum_{i=1}^{t} r_i w_i = \sum_{i=1}^{t} \delta_i(y) w_i$. For each $i \in I$, put $f_i = \delta_i \circ g : M \to R$. We claim that $\{m_i \mid i \in I\}$ and $\{f_i \mid i \in I\}$ satisfy conditions in (3). Let m

$=\sum_{j=1}^{k} s_j m_j$, $s_j \in$ R. $f_i(m) = \delta_i \circ g(m) \neq 0$ for any finitely many $i \in I$ and $\sum_{i\in I} f_i(m)m_i - m = \sum_{i\in I} \delta_i \circ g(m)m_i - m = \sum_{i\in I} \delta_i(g(m))\psi(w_i) - m = \psi(\sum_{i\in I} \delta_i(g(m))w_i - m = \psi(\sum_{i\in I} \delta_i(\sum_{j=1}^{k} \delta_j g(m)w_j)w_i) - m = \psi(\sum_{j=1}^{k} \delta_j g(m)w_j)) - m = \psi(g(m)) - m \in T$

(3) ⟹ (1) Let

$$0 \longrightarrow K \xrightarrow{i} W \xrightarrow{\pi} M \longrightarrow 0$$

be short exact sequence where W is a free R-module and K = ker(π). Let $\{y_i \mid i \in I\}$ be a basis for W. Then $\{\pi(y_i) \mid i \in I\}$ is a generated set of M. By (3) there exists family $\{f_i \mid i \in I\} \subseteq M^*$ such that (a) and (b) in (3) are satisfied. Define $\theta : W \to W$ by $\theta(x) = \sum_{i\in I} f_i(a)y_i$ where $\pi(x) = a$ for each x in W. Then we have $\pi\circ\theta(x) - \pi(x) = \pi(\sum_{i\in I} f_i(a)y_i) - \pi(x) = \sum_{i\in I} f_i(a)m_i - a \in T$.It is clear that $\ker(\pi) \subseteq \ker(\theta)$. Therefor by Theorem (3.3). M is (T)-projective.

Example 3.9: Let T be an R-module which is not projective and $M = T\oplus H$ where $H = \oplus R$ is a direct sum of a countable number of copies of R. It is clear that M is not projective. We claim that M is (T)-projective Let $\{x_i \mid i \in N\}$ be a basis for H. Then $T \cup \{x_i \mid i \in N\}$ is a generated set of M. For each $i \in N$, define $fi : H \to R$ by $f_i(x_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

and fi can be extended by linearity to all H, therefore if $x = \sum_{i=1}^{n} r_i x_i$, then $f_i(x) = r_i$. Again f_i can be extended to $\acute{f}_i : M \to R$ by putting $\acute{f}_i(y) = 0$ for each $y \in T$. Then $\{\acute{f}_i \mid i \in N\} \subseteq M^*$. Now, let $m \in M$, then $m = y + x$ where $y \in T$ and $x \in H$. $\acute{f}_i(m) = \acute{f}_i(y + \sum_{i=1}^{n} r_i x_i) = r_j$, hence $\sum_{j=1}^{n} \acute{f}_j(m)x_j - m = \sum_{j=1}^{n} r_j x_j - m = x - m \in T$. Thus Theorem (3.8) shows that M is (T)-projective.

Proposition 3.10: Let M be a projective R-module and N a (T)-pure submodule(hence (T)-direct summand) of M. Then N is (T∩N)-projective.

Proof: Let n ∈ N . By dual basis lemma for projective modules, there are $\{m_i \mid i \in I\} \subseteq M$ and $\{\varphi_i \mid i \in I\} \subseteq M^*$ such that $\varphi_i(n) \neq 0$ for only finitely many i ∈ I and $n = \sum_{i=1}^{t} \varphi_i(n)m_i$. (T)-purity of N implies that there exist $n_i \in N$ such that $n - \sum_{i=1}^{t} \varphi_i(n)n_i \in T \cap N$. Put $\gamma_i = \varphi_i|_N$, then $n - \sum_{i=1}^{t} \gamma_i(n)n_i \in T \cap N$. Thus Theorem (3.8) shows that N is (T ∩ N)-projective.

Corollary 3.11: Pure submodule of projective module is projective.

It is known that if M is a projective R-module then J(M) = J(R)M whereJ(M) is the Jacobson radical of M [4]. Now, we are in a position to obtain the following result which is a generalization of that for projective modules.

Proposition 3.12: Let M be a (T)-projective R-module and T a small submoduleof M. Then J(M) = J(R)M + T.

Proof: Let x ∈ J(M). Then there are $\{y_i \mid i \in I\} \subseteq M$ and $\{\varphi_i \mid i \in I\} \subseteq M^*$ such that $x - \sum_{i=1}^{t} \varphi_i(x)y_i \in T$, but $\varphi_i(J(M)) \subseteq J(R)$. ThenJ(M) ⊆J(R)M + T. As J(M) is the sum of all submodules of M, thusJ(M) = J(R)M + T

The next theorem is a partial converse of Proposition (2.4).

Theorem 3.13: Let M be a projective R-module. If N is a finitely generated (T)-pure submodule of M, then N is (T)-direct summand

Proof: let M be a free R-module with basis $\{e_j \mid j \in J\}$ and $\{x_1, x_2, \ldots, x_m\}$ a generated set of N. Then $x_i = \sum_{j \in J_0} a_{ij} e_j$ for some $a_{ij} \in R$ and finite subsetJ_o of J,(i = 1, 2, . . . ,m). (T)-purity of N implies that there are $n_j \in N (j \in J_o)$such that $x_i - \sum_{j \in J_0} a_{ij} n_j \in N \cap T$. If θ : N → M is the inclusion map, then define α : M → N by $\alpha(e_j) = \begin{cases} n_j & \text{if } j \in J_0 \\ 0 & \text{if } j \notin J_0 \end{cases}$

Then we have $\alpha \circ \theta(x) - x \in T$ for each $x \in N$ and so N is (T)-direct summand. For the general case, there exists a free R-module W such that M is direct summand in W, hence M is pure in W, so N is (T)-pure in W. By the particular case, N is (T)-direct summand of W. Thus there is a submodule V ofW such that W = N+V and $N \cap V \subseteq T$. This implies that $M = N+(M \cap N)$ and $N \cap (M \cap V) \subseteq T$ This shows that N is a (T)-direct summand of M.

Next, we give a criteria of purity relative to a submodule.

Theorem 3.14: Let N and T be two submodules of an R-module M. If N is (T)-projective and every R-homomorphism $N \rightarrow R$ can be extended to all M, then N is (T)-pure in M.

Proof: Let $x_i = \sum_{i=1}^{n} a_{ij} y_i \in N$ where $y_j \in M$ and $a_{ij} \in R$ (i = 1, 2, . . . ,m). Theorem (3.8): implies that there are a set of generators {eh | h ∈ H} of N and a set $\{f_h \mid h \in H\}$, $f_h \in N^*$ such that for each $x \in N$, $f_h(x) \neq 0$ for only finitely many $h \in H$ and $x - \sum_{h \in H} f_h(x) e_h \in T$. By hypothesis, f_h extends to $g_h : M \rightarrow R$ for each $h \in H$. So $f_h(x_i) = \sum_{h \in H} a_{ij} g_h(y_j)$ $\forall h \in H$. Then for each i = 1, 2, . . . ,m, there exists $v_i \in T$ such that $x_i = \sum_{h \in H} f_h(x_i)$ $e_h + v_i = \sum_{j=1}^{n} a_{ij} \sum_{h \in H} g_h(y_i)$ $e_h + v_i$and N is (T)-pure in M.

LetM and U be two R-modules. U isM-injective if for given R-monomorphism $f : K \rightarrow M$, each R-homomorphism $g : K \rightarrow U$ can be extended to an R-homomorphism $h : M \rightarrow U$. An R-module M is called injective if M isU-injective for each R-module U. A ring R is self-injective if R is R-injective.

Let M be an R-module. Then the condition R is M-injective includes the extension condition of Theorem (3.14). Hence we have the following corollaries.

Corollary 3.15: Let M be an R-module and R is M-injective. Then every (T)-projective submodule of M is (T)-pure.

Corollary 3.16: Let R be a left self-injective ring. Then every (T)-projective submodule of any R-module is (T)-pure.

An R-module M is quasi-continuous if for every two submodules U and V of M with $U \cap V = 0$, there exists an R-endomorphism θ of M such that $U \subseteq \ker(\theta)$ and $V \subseteq \ker(1 - \theta)$ [7]. It is known that if $M = U \oplus V$ is quasi-continuous R-module, then U is V -injective and V is U-injective [5].

Corollary 3.17: Let M be an R-module. If M_R is quasi-continuous, then M is injective and every (T)-projective submodule of M is (T)-pure.

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