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Certain Subclasses Of Meromorphically Multivalent Functions Involving New Linear Operator

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Abstract

The main object of this article is to study the subclasses of multivalent meromorphic functions by using a linear operator in punctured unit disk. Some geometric properties like coefficients bound, distortion theorems and convolutions property
are investigated.

Key Words : Multivalent function, Starlike and Convex function, Hadamard product, . meromorphic function

عدد محدد من صفوف الدوال المتعددة التكافؤ الميرومورفيك باستخدام مؤثر خطي

الملخص

الهدف الرئيسي في هذا البحث هو دراسة عائلة من الدوال متعددة التكافؤ الميرومورفيك باستخدام مؤثر خطي المعرف على دائرة الوحده المثقبه بالاضافه الى بعض الخواص الهندسية مثل حدود المعاملات ونظرية التحريف بالاضافة الى مفهوم الالتفاف (ضرب هادمرد)

1. Introduction :

Let $\sum_p$ denots the clas of functions of the form :

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_k z^{k-p}, (p \in N) \qquad \ldots\ldots. \quad 1\text{-}1$$

which are analytic and p-valent in the punctured unit disk U_ = {z : 0 < |z| < 1}.

We denote by $\sum S_P^*(\alpha)$ *and* $\sum C_P(\alpha)$ the subclass of meromorphic p-valent starlike and convex functions,respectively,that is,

$$\sum s_p^*(\alpha) = \left\{ f(z) \in \sum_p : \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) \prec -\alpha \, (0 \le \alpha \prec p) \right\}, \qquad \ldots.. \ 1-2$$

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$$\sum c_p(\alpha) = \left\{ f(z) \in \sum_p : 1 + \left(\frac{zf''(z)}{f'(z)} \right) \prec -\alpha \, (0 \le \alpha \prec p) \right\}, \quad \ldots.. 1-3$$

For a function $f(z) \in \sum_p$,we define the following operator which studied by El- Ashwah[7].

$$I_p^m(\lambda,\ell) f(z) = z^{-p} + \sum_{k=1}^{\infty} (\frac{\ell+\lambda k}{\ell})^m a_k z^{k-p}, (\lambda \succ 0; \ell \succ 0); m \in N_0; z \in u) \quad . 1\text{- } 4$$

We note that:

1- $I_1^m(1,\ell) f(z) = I(m,\ell) f(z)$, studied by Cho. et. al. [6].

2- $I_p^m(1,\ell) f(z) = D_P^M f(z)$, studied by Aouf and Hossen[4],Liu and Owa[8] and Sirvistava and Patel[12].

3- $I_p^m(1,\ell) f(z) = I^m f(z)$, studied by Uralgaddi and Somanatha [14].

4- $I_1^m(\lambda,\ell) f(z) = D_\lambda^m f(z)$, studied by Al-Oboudi and Al-Zkeri[1].

<u>Definition 1.1</u>.A function f is in the Class $M_{\ell,p}^m(\alpha,\beta,\lambda,\gamma)$ if it satisfies the following condition:

$$\left| \frac{z^{p+1}(I_p^m(\lambda,\ell) f(z))' + p}{(2\gamma-1) z^{p+1}(I_p^m(\lambda,\ell) f(z))' + (2\gamma\alpha - p)} \right| < \beta, z \in u^*$$

Where $(0 \le \alpha \prec p); \lambda \succ 0; \ell \succ 0; m \in N_0; Z \in U; \frac{1}{2} \le \gamma \le 1 \, and \; 0 < \beta \le 1,$

. There are many special cases of this class were studied by several authors (for example see Cho et.al.[5] and Aouf [2]). Also,several results were obtained and studied for subclasses of multivalent meromorphic functions by many authors (see e.g.[9],[10],[3]and [13]). In the present paper we have obtained coefficient bounds , distortion bounds and some properties of convolutions of functions in this class .

2. <u>Coefficient Bounds</u> :

In this section,necessary and sufficient condition for a function f(z) belongs to the class $M_{1,p}^m(\alpha,\beta,\lambda,\gamma)$ are obtained.

Theorem 2.1. A function f(z) of the form (1.1) is in the class $M_{1,p}^{m}(\alpha,\beta,\lambda,\gamma)$ if and only if :

$$\sum_{k=1}^{\infty}\frac{(k-p)(\frac{\ell+\lambda k}{\ell})^{m}(1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)}\leq 1. \qquad +\ldots\ldots\ldots\ldots \quad 2-1$$

Proof : Suppose (2.1) hold true.Then we obtain :

$$\left|z^{p+1}(I_p^m(\lambda,\ell)f(z))'+p\right|-\beta\left|(2\gamma-1)z^{p+1}(I_p^m(\lambda,\ell)f(z))'+(2\gamma\alpha-p)\right|\prec 0.$$

There for :

$$\left|\sum_{k=1}^{\infty}(k-p)(\frac{\ell+\lambda k}{\ell})a_k z^k\right|-\beta\left|2\gamma(p-\alpha)-\sum_{k=1}^{\infty}(k-p)(\frac{\ell+\lambda k}{p})^m a_k z^p\right|\prec 0.$$

For |z| = r < 1 the left hand side of last inequality is bounded above by :

$$\sum_{k=1}^{\infty}(k-p)(\frac{\ell+\lambda k}{p})a_k z^k-\beta(2\gamma(p-\alpha)-\sum_{k=1}^{\infty}(k-p)(\frac{\ell+\lambda k}{p})^m a_k z^p$$

$$\prec\sum_{k=1}^{\infty}(k-p)(\frac{\ell+\lambda k}{p})^m(1+\alpha\beta\gamma-\beta)a_k-2\beta\gamma(p-k)\leq 0.$$

Then :

$$f(z)\in M_{1,P}^{M}(\alpha,\beta,\lambda,\gamma).$$

For converse,

Let $f(z)\in M_{1,P}^{M}(\alpha,\beta,\lambda,\gamma).$ be given by (1.1).Then :

$$\left|\frac{z^{p+1}(I_p^m(\lambda,\ell)f(z))'+p}{(2\gamma-1)z^{p+1}(I_p^m(\lambda,\ell)f(z))'+(2\gamma\alpha-p)}\right|$$

$$=\left|\frac{\sum_{k=1}^{\infty}(k-p)(\frac{\ell+}{\ell})^m a_k z^k}{(2\gamma-1)(p-\sum_{k=1}^{\infty}(k-p)(\frac{\ell+\lambda k}{\ell})^m a_k z^k-(2\gamma\alpha-p)}\right|.$$

Since |Re(z)| _ |z| for all z.Then we get :

$$. \operatorname{Re} \frac{\sum_{k=1}^{\infty}(k-p)(\frac{\ell+\lambda k}{\ell})^m a_k z^k}{(2\gamma(p-\alpha)-\sum_{k=1}^{\infty}(2\gamma-1)(p-k)(\frac{\ell+\lambda k}{\ell})^m a_k z^k} \prec \beta \qquad \text{...... 2-2}$$

Choose values of z on the real axis so that $(z^{p+1}(I_p^m(\lambda,\ell)f(z))'$ is real Then , Upon clearing the denominator in(2.2) and letting : $z \to 1-$through real values , we obtain

$$\sum_{k=1}^{\infty}(k-p)(\frac{\ell+\lambda k}{\ell})(1+2\beta\gamma-\beta)a_k\, 2\beta\gamma(p-\alpha).$$

This gives required condition.

<u>**Corollary 2.2**</u>: Let f(z) of the form (1.1) is in the class $M_{\ell,p}^m(\alpha,\beta,\lambda,\gamma)$

Then :

$$a_k \le \frac{2\beta\gamma(p-\alpha)}{(k-p)(\frac{\ell+\lambda k}{\ell})^m(1+2\beta\gamma-\beta)}(k\ge 1) \qquad \text{....... } 2-3\,.$$

$$f(z)=z^{-p}+\frac{2\beta\gamma(p-\alpha)}{(k-p)(\frac{\ell+\lambda k}{\ell})^m(1+2\beta\gamma-\beta)}z^{k-p}(k\ge 1) \qquad .\,2-4\,.$$

The result is sharp of the function :

<u>3. Distortion Theorem :</u>

<u>**Theorem 3.1**</u>.If a function f(z)defined by (1.1) is in the class $M_{\ell,p}^m(\alpha,\beta,\lambda,\gamma)$), then :

$$T^{-p}-\frac{2\beta\gamma(p-\alpha)}{(k-p)(\frac{\ell+\lambda k}{\ell})^{m}(1+2\beta\gamma-\beta)}T^{1-P}\leq\left|I_{p}^{m}(\lambda,\ell)f(z)\right|$$

$$\leq T^{-p}+\frac{2\beta\gamma(p-\alpha)}{(k-p)(\frac{\ell+\lambda k}{\ell})^{m}(1+2\beta\gamma-\beta)}T^{1-P}, \qquad \ldots\ldots\ 3-1.$$

For sharpness, take the functions f(z) defined by :

$$f(z)=z^{-p}+\frac{2\beta\gamma(p-\alpha)}{(k-p)(\frac{\ell+\lambda k}{\ell})^{m}(1+2\beta\gamma-\beta)}z^{1-p} \qquad \ldots.\ 3-2$$

Proof : By Theorem 2.1,we obtain :

$$(1-p)(1+2\beta\gamma-\beta)\sum_{k=1}^{\infty}a_{k}\leq(k-p)(\frac{\ell+\lambda k}{\ell})^{m}(1+2\beta\gamma-\beta)a_{k}$$

$$\leq 2\beta\gamma(p-\alpha),$$

therefore,

$$\sum_{k=1}^{\infty}a_{k}\leq\frac{2\beta\gamma(p-\alpha)}{(k-p)(\frac{\ell+\lambda k}{\ell})^{m}(1+2\beta\gamma-\beta)}$$

Thus, for 0 < |z| = r < 1,

$$\left|I_{pt}^{m}(\lambda,\ell)f(z)\right|\leq T^{-p}+\sum_{k=1}^{\infty}(\frac{\ell+\lambda+k}{p})^{m}a_{k}T^{k-p}$$

$$\leq T^{-p} + T^{1-p}\left(\frac{\ell+\lambda}{\ell}\right)^m \sum_{k=1}^{\infty} a_k$$

$$\leq T^{-p} + \frac{2\beta\gamma(p-\alpha)}{(1-p)\left(\frac{\ell+\lambda}{\ell}\right)^m (1+2\beta\gamma-\beta)} T^{1-p}$$

And :

$$\left|I_p^m(\lambda,\ell) f(z)\right| \geq T^{-p} - \sum_{k=1}^{\infty}\left(\frac{\ell+\lambda+k}{p}\right)^m a_k T^{k-p}$$

$$\geq T^{-p} - T^{1-p}\left(\frac{\ell+\lambda}{\ell}\right)^m \sum_{k=1}^{\infty} a_k .$$

$$\geq T^{-p} - \frac{2\beta\gamma(p-\alpha)}{(1-p)\left(\frac{\ell+\lambda}{\ell}\right)^m (1+2\beta\gamma-\beta)} T^{1-p} .$$

This gives the required result.

<u>Corollary 3.2</u>. If a function f(z)defined by (1.1) is in the class $M_{\ell,p}^m(\alpha,\beta,\lambda,\gamma)$

Then :

$$\left(T^{-p} - \frac{2\beta\gamma(p-\alpha)}{(1-p)\left(\frac{\ell+\lambda}{\ell}\right)^m (1+2\beta\gamma-\beta)} T^{1-p}\right) \leq |f(z)|$$

$$\geq \left(T^{-p} + \frac{2\beta\gamma(p-\alpha)}{(1-p)\left(\frac{\ell+\lambda}{\ell}\right)^m (1+2\beta\gamma-\beta)} T^{1-p}\right) \qquad .. 3\text{-}3 .$$

4. Hadamard Product

Let g(z) $\in \sum p$ defined by **:**

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_k z^{k-p}.(p \in N) \quad \ldots\ldots\ldots\ldots.. \quad 4-1 .$$

Then the hadamard product (convolution) of the functions f(z) and g(z) denoted by f *g and defined:

$$(f * g)(z) = z^{-p} + \sum_{k=1}^{\infty} a_k b_k z^{k-p}.(p \in N) \quad \ldots\ldots\ldots \quad 4-2 .$$

Theorem 4.1. Let the functions f, g defined by (1.1) and (4.1) respectively, be in the class $M_{\ell,p}^{m}(\alpha,\beta,\lambda,\gamma)$.

Then $(f * g) \in M_{\ell,p}^{m}(\alpha,\beta,\lambda,\gamma)$, where :

$$\sigma = p - \frac{2\beta\gamma\,(p-\alpha)^2}{(1-p)(\frac{\ell+\lambda}{\ell})^m (1+2\beta\gamma-\beta)}.$$

$$\geq (T^{-p} + \frac{2\beta\gamma\,(p-\alpha)}{(1-p)(\frac{\ell+\lambda}{\ell})^m (1+2\beta\gamma-\beta)} T^{1-p})$$

Proof : By using the technique used earlier by Schild and Silverman[11], in order to prove this theorem ,we need to find the largest σ such that :

$$\sum_{k=1}^{\infty} \frac{(k-p)(\frac{\ell+\lambda k}{\ell})^m (1+2\beta\gamma-\beta)}{2\beta\gamma(p-\sigma)} a_k \leq 1. \quad \ldots\ldots \quad 4-3 .$$

Since $f, g \in M_{\ell,p}^{m}(\alpha,\beta,\lambda,\gamma)$,we see that :

$$\sum_{k=1}^{\infty} \frac{(k-p)(\frac{\ell+\lambda k}{\ell})^{m}(1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)} a_k \leq 1.$$

And ,

$$\sum_{k=1}^{\infty} \frac{(k-p)(\frac{\ell+\lambda k}{\ell})^{m}(1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)} b_k \leq 1.$$

Therefore, by the Cauchy Schwarz inequality, we get :

$$\sum_{k=1}^{\infty} \frac{(k-p)(\frac{\ell+\lambda k}{\ell})^{m}(1+2\beta\gamma-\beta)}{2\beta\gamma(p-\sigma)} \sqrt{a_k b_k} \leq 1.$$

Thus, we need only to show that :

$$\frac{a_k b_k}{p-\sigma} \leq \frac{\sqrt{a_k b_k}}{p-\alpha}, (k \geq 1). \qquad \text{Or}$$

$$\sqrt{a_k b_k} \leq \frac{p-\sigma}{p-\alpha}, (k \geq 1).$$

Now, by Theorem 1,its sufficient to show that :

$$\frac{2\beta\gamma(p-\alpha)}{(k-p)(\frac{\ell+\lambda k}{\ell})(1+2\beta\gamma-\beta)} \leq p-\sigma p-\alpha, \qquad \ldots\ldots.. \quad 4-4$$

It follows from (4-4) that :

$$\sigma \leq p - \frac{2\beta\gamma(p-\alpha)^2}{(k-p)(\frac{\ell+\lambda k}{\ell})(1+2\beta\gamma-\beta)}.(k \geq 1),$$

Hence, define $\psi(k)$ by :

$$\psi(k) = p - \frac{2\beta\gamma(p-\alpha)^2}{(k-p)(\frac{\ell+\lambda k}{\ell})(1+2\beta\gamma-\beta)} . (k \geq 1),$$

Its clear that $\psi(k)$ is an increasing function of k.Thus,we obtain :

$$\sigma \leq \psi(1) = p - \frac{2\beta\gamma(p-\alpha)^2}{(1-p)(\frac{\ell+\lambda k}{\ell})(1+2\beta\gamma-\beta)} . (k \geq 1),$$

This completes the proof.

Theorem 4.2: Let the functions f, g defined by (1.1) and (4.1) respectively, be in the class $M^m_{\ell,p}(\alpha,\beta,\lambda,\gamma)$ with $|b_k| \leq 1$, k = 1, 2, 3, ...p $\in$N.Then : $f,g \in M^m_{\ell,p}(\alpha,\beta,\lambda,\gamma)$.

Proof. Since :

$$\sum_{k=1}^{\infty} \frac{(k-p)(\frac{\ell+\lambda k}{\ell})^m (1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)} |a_k b_k|$$

$$\sum_{k=1}^{\infty} \frac{(k-p)(\frac{\ell+\lambda k}{\ell})^m (1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)} a_k |b_k|$$

$$\leq \sum_{k=1}^{\infty} \frac{(k-p)(\frac{\ell+\lambda k}{\ell})^m (1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)} a_k \leq 1 .$$

therefore by Theorem 2.1 ,we have the result and the proof is complete.

Theorem 2.4: Let the functions f, g defined by (1.1) and (4.1) respectively, be in the class $M^m_{\ell,p}(\alpha,\beta,\lambda,\gamma)$ and :

$$(1-p)\left(\frac{\ell+\lambda}{\ell}\right)^{m}(1+2\beta\gamma-\beta)-4\beta\gamma(p-\alpha)\geq 0,$$

Then the function h(z) defined by :

$$h(z)=z^{-p}+\sum_{k=1}^{\infty}(a_k^2+b_k^2)z^{k-p}.$$

belongs to the class $M_{\ell,p}^{m}(\alpha,\beta,\lambda,\gamma)$.

Proof. In virtue of Theorem 2.1 and since $f(z)\in M_{\ell,p}^{m}(\alpha,\beta,\lambda,\gamma)$ we get :

$$\sum_{k=1}^{\infty}\frac{(k-p)\left(\frac{\ell+\lambda k}{\ell}\right)^{m}(1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)}a_k\leq 1.$$

So :

$$\sum_{k=1}^{\infty}\left[\frac{(k-p)\left(\frac{\ell+\lambda k}{\ell}\right)^{m}(1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)}\right]^{2}a_k^2\leq 1.$$

Also,since $g(z)\in M_{\ell,p}^{m}(\alpha,\beta,\lambda,\gamma)$, we get :

$$\sum_{k=1}^{\infty}\left[\frac{(k-p)\left(\frac{\ell+\lambda k}{\ell}\right)^{m}(1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)}\right]^{2}b_k^2\leq 1.$$

Therefore,

$$\frac{1}{2}\sum_{k=1}^{\infty}\left[\frac{(k-p)(\frac{\ell+\lambda k}{\ell})^{m}(1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)}\right]^{2}(a_k^2+b_k^2)\ \le 1.$$

Then, we must show that :

$$\sum_{k=1}^{\infty}\left[\frac{(k-p)(\frac{\ell+\lambda k}{\ell})^{m}(1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)}\right]^{2}(a_k^2+b_k^2)\ \le 1.$$

Thus it will be satisfies if,

$$\frac{(k-p)(\frac{\ell+\lambda k}{\ell})^{m}(1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)}$$

$$\le \frac{1}{2}\sum_{k=1}^{\infty}\left[\frac{(k-p)(\frac{\ell+\lambda k}{\ell})^{m}(1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)}\right]^{2}.$$

Or :

$$(k-p)(\frac{\ell+\lambda k}{\ell})^{m}(1+2\beta\gamma-\beta)-4\beta\gamma(p-\alpha)\ge 0.$$

For k = 1, 2, 3, ... the left hand side of the last inequality is increasing function of k, and its satisfied for all k if :

$$(1-p)(\frac{\ell+\lambda}{\ell})^{m}(1+2\beta\gamma-\beta)-4\beta\gamma(p-\alpha)\ge 0.$$

and this completes the proof of theorem.

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