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Locally Split Homomorphisms Relative To A Sub module

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Abstract:

In this article,for a positive integer n ,the concept of n-locally split homomorphism relative to a sub module has been introduced and studied,which will turn out to be most useful in the studying and providing characterizations of local projectivity, local-regularity in the sense of (Zelmanowitz, Field house and Ware) relative to a sub module.They present generalization of projectivity and the three types of regularity which have been mentioned respectivily.

Keywords: n-locally(T)-split homomorphisms,n-locally(T)-projective modules, n-(T)-regular modules,(T)-pure sub modules and Zelmanowitz (Field house) (T)-regular modules.

الملخص

في هذا البحث ،لكل عدد صحيح موجب n ،قدمنا فكرة التشاكل المفصول محليا ضمن النمط –n بالنسبة للمقاسات الجزئية،لدراسة وإعطاء تميزا للإسقاط المحلي،والانتظام المحلي بمفهوم (Zelmanowitz, Field house and Ware) بالنسبة للمقاسات الجزئية.وقدمنا تعميما للاسقاطية وثلاث أنواع من الانتظام.

1- Introduction

Nowadays, there are three possible generalizations of the notion of Von Neumann regular rings to the general module theoretic setting by Zelmanowitz [9], and Field house [5], as well as Ware [8], each one called regular.

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A right R-module M is Zelmanowitz-regular, if for each x∈M, there exists an R-homomorphism α ∈M* = HomR(M,R) satisfies x = xα(x). The Field house-regular module

Was defined as one whose sub modules are pure, while Ware-module was defined as a projective module in which every principal sub module is direct summand. The concept of locally split homomorphisms was introduced in [3].Let M and N be R-modules, and α : N → M an R-homomorphism. A is called locally split, if for each xo∈ α(N), there is an R-homomorphism β : M → N such that α(β(xo)) = xo. This concept had been utilized to characterize Zelmanowitz-regula modules, and modules in which every sub module is locally split, that is, the inclusion mapping i : N → M is locally split for each sub modules N of M.

Many algebraic structures had been restudied relative to a class of sub modules, as semi-regular modules relative to a fully invariant sub module [2], uniform extending modules [4], quasi-injective modules relative to the closed sub modules class [7], pseudo-injective modules relative to a principal sub modules class [10].Recently, projective module relative to a sub module has been studied in [1].Let P be an R-module and T a sub module of P. P is called (T)-projective , if for every R-epimorphismf : A →B and R-homomorphism g : P → B, there exists an R-homomorphism h : P → A such that foh(x) − g(x) ∈g(T) for all x∈P.

In section two of this work we introduce the concept of n-locally split R-homomorphism relative to a sub module. Several properties have been given and considered modules in which the inclusion mapping of every sub module is locally split with respect to a sub module. In section three, we utilize locally split homomorphism relative to a sub module to characterize locally(T)-projective modules which is a generalization of (T)-projective modules [1]. Many properties and characterizations have been investigated. Finally, in section four, we introduce Zelmanowitzregularand Field house-regular modules relative to a sub module, and characterize them in terms of our notions in section two, and we show that the three notions of regularity relative to a sub module are coinciding under locally (T)-projective modules.

In what follows, R will represent an associative ring with identity and R-moduleM will mean unitary right R-module, unless otherwise stated.

2. n-Locally Split Homomorphisms Relative To A Sub module

Splitting homomorphisms are valuable tools for splitting modules into internal direct sum. In this section we introduce a generalization of locally split homomorphisms.

Definition 2.1: Let M , N be two R-modules and T a sub module of M. An R-homomorphism $\alpha : N \to M$ is called n-locally (T)-split, if for any finite number of $x1, x2, ..., xn \in \alpha(N)$, there exists an R-homomorphism $\beta: M \to N$ such that $\alpha(\beta(xi)) - xi \in T$ for each i = 1, 2, ..., n.

This concept extends some notions in the literature. It is clear that an Rhomomorphism $\alpha : N \to M$ is n-locally (0) -split if and only if it is locally split which was introduced in [3]. Clearly, an R-homomorphism is n-locally (T)-split if and only if it is k-locally (T)-split for all $k \leq n$. It is well-known that, the Z-module Z is indecomposable and hence the inclusion mapping i : $2Z \to Z$ is not locally split, but it is 1-locally (6Z)-split, since if we consider the Z-homomorphism $\alpha : Z \to 2Z$ defined by $\alpha(x) = 4x$ for each $x \in Z$, then for each $y \in 2Z$ we have $\alpha(y) - y \in 6Z$

Let M be an R-module and N, T be sub modules of M. N is called n-locally(T)-split, if the inclusion mapping $i : N \to M$ is n-locally (T)-split, that is, for any finite number of $x1, x2, ..., xn \in N$, there exists an R-homomorphism $s : M \to N$ such that $s(x_i)-x_i \in T$ for each $i = 1, 2, ..., n$. the notion of n-locally (0)-split sub moduleswas introduced by Ramamurthi and Rangaswany [6] by the name of strongly pure sub modules.

Recall that a sub module N of an R-module M is fully invariant if $\alpha(N) \subseteq N$ for each R-endomorphism α of M. It is known that an R-homomorphism $\alpha : A \to M$ is n-locally (0)-split if and only if it is 1-locally(0)-split[3]. Relative to a non-zero sub module we have the following:

Proposition 2.2: Let A and B be R-modules and T a fully invariant sub moduleof B.Then an R-homomorphism $\alpha : A \to B$ is n-locally (T)-split if and only if it is1-locally(T)-split.

Proof: The "only if" part is clear for any arbitrary sub module T of B. We shall use induction to prove the "if" part. Suppose that our statement is true for n-1 where $n > 1$.Then there exists an R-homomorphism $\beta_1 : B \to A$ such that $x_i - \alpha(\beta_1(x_i)) \in T$ for each $i = 1, 2, ..., n - 1$. As $x_n - \alpha(\beta_1(x_n)) \in \alpha(A)$,there exists an Rhomomorphism $\beta_2 : B \to A$ such that $x_n-\alpha(\beta_1(x_n))-\alpha(\beta_2(x_n-\alpha(\beta_1(x_n)))) \in T$ .Put $\beta = \beta_1 + \beta_2 - \beta_2 \circ \alpha \circ \beta_1$, then $\beta : B \to A$ and $x_n-\alpha(\beta(x_n)) = x_n- \alpha(\beta_1(x_n)) - \alpha(\beta_2(x_n)) + \alpha(\beta_2(\alpha(\beta_1(x_n))))$ $= x_n-\alpha(\beta_1(x_n)) - \alpha(\beta_2(x_n - \alpha(\beta_1(x_n)))) \in T$.

Furthermore, for $i = 1, 2, ..., n$, we have $x_i - \alpha(\beta(x_i)) = x_i - \alpha(\beta_1(x_i)) - \alpha(\beta_2(x_i))+\alpha(\beta_2(\alpha(\beta_1(x_i))))= x_i - \alpha(\beta_1 x_i)- \alpha(\beta_2(x_i - \beta_1(x_i))) = v - \alpha \circ \beta_2(v) \in T$ where $v = x_i - \alpha(\beta_1(x_i))$.This shows that,$x_i - \alpha(\beta(x_i))$_T for each $i = 1, 2, ..., n$ and hence α is n-locally(T)-split.

According to the above proposition, all results follow will doing either in thesense of (1−locally) or (n−locally) concept for arbitrary sub module,these results will be true in the sense of (n − locally) or (1 − locally) concept respectively for fully invariant sub module, unless otherwise mentioned.

Proposition 2.3: Let h : A → B be an R-homomorphism, h̀be the R-epimorphismfrom A onto h(A) and T be a fully invariant sub module of h(A). Then the following are equivalent:

(1) h is n-locally (T)-split,

(2) h̀ is n-locally (T)-split and h(A) is n-locally (T)-split sub module in B.

Proof: Let x1, x2, ..., xn be elements in h(A). Assume (1), then there exists an R-homomorphism q : B →A such that h(q(xi)) − xi_T for each i = 1, 2, ..., n. Let s = h ∘ q : B → h(A) such that s(xi) − xi_T and hence h(A) is n-locally (T)-split in B. If we denote q̀ = q|h(A) : h(A) →A, then h̀ (q̀(xi)) − xi∈T foreach i = 1, 2, ..., n which shows that h0 is n-locally (T)-split. Assume (2), then there are R-homomorphism s : B → h(A) with s(xi) − xi∈T and q̀ : h(A) → A with h̀ (q̀ (xi)) – xi∈T for each i = 1, 2, ..., n. Let q = q̀ ∘ s : B → A. Thenh(q(xi))− xi = h̀ (q̀ (s(xi)))−xi = h̀ (q̀(xi+ti))−xi = h̀ (q̀ (xi))−xi+ h̀(q̀ (ti)) ∈Tfor each i = 1, 2, ..., n, and some ti_T . Thus h is n-locally(T)-split.

The following corollary follows directly from proposition (2.3) and proposition (2.2).

Corollary 2.4:Let A,B, h, h̀andT be as in proposition (2.3) . Then the following are equivalent:

(1) h is 1-locally (T)-split

(2) h̀ is 1-locally (T)-split and h(A) is 1-locally (T)-split sub module in B.

Let M be an R-module and T a sub module of M. A sub module N of M is called (T)-pure, if MA ∩ N = NA + T ∩ (MA ∩ N) for each right ideal A of R.This is equivalent to saying that, for every finite sets {mi} ⊆ M,{nj} ⊆ N and {rij} ⊆ R with nj =$\sum_{i=1}^{n} m_i r_{ij}$,j = 1, 2, ...,m, there is a finite set {xi} ⊆ N suchthat nj $-\sum_{i=1}^{n} x_i r_{ij} \in N \cap T$ for each j = 1, 2, ...,m [1]. In the Z-module Z,2Z is (6Z)-pure sub module in Z which is not pure.

Proposition 2.5: Let M be an R-module and T a sub module of M. Then

(1) Every 1−locally (hence n−locally) (T)-split sub module in M is (T)-pure.

(2) Let h : M → M̀be R-epimorphism. If h is 1 − locally(h(T)) − split, then Ker(h) is 1 − locally (T)-split in M.

Proof: Let N be 1 − locally (T)-split sub module of M and nj = $\sum_{i=1}^{m} x_i r_{ij}$ {nj} ⊆ N, j = 1, 2, ..., n, {xi} ⊆ M and {rij} ⊆ R. For each j = 1, 2, ..., n, there exists an R-homomorphism sj: M → N such that sj(nj)−nj∈ T . Put s $=\sum_{j=1}^{n} s_j$, then s : M → N. Hence s(xi) ∈ N and $\sum_{i=1}^{m} s(x_i) r_{ij}$ −nj $=\sum_{i=1}^{m} (\sum_{j=1}^{n} s_j)(x_i) r_{ij} - \sum_{i=1}^{m} x_i r_{ij} = \sum_{j=1}^{n} (\sum_{i=1}^{m} s_j(x_i) - xi)rij = \sum_{j=1}^{n} s_j - nj \in T$. This shows that N is (T)-pure. For the second statement, let h : M → M̀be an R-homomorphism and nj = $\sum_{i=1}^{m} x_i r_{ij}$ □ Ker(h). Then$\sum_{i=1}^{m} h(x_i) r_{ij} = 0$. Since h(xi) ∈ h(M) = M̀ there is an R-homomorphism q : M̀→ M such that h(q(h(xi)))−h(xi) = h(ti) for some ti∈ T and each i = 1, 2, ...,m, and hence ti+xi−q(h(xi))□ $\sum_{i=1}^{m} t_i$+xi−q(h(xi)))rij−nj = $\sum_{i=1}^{m} t_i r_{ij}$ □ T .This shows that Ker(h) is (T)-split in M.

Corollary 2.6:Let h : A → B be an R-homomorphism and T be a fully invariant sub module of h(A). If h is 1 − locally(T)-split homomorphism, then h(A) is a (T)-pure sub module of B.

We call an R-module M, n−(T)−regular, if each sub module of M is n−locally(T)-split, where T is a sub module of M.

It is clear that, if M is n−(T)−regularR−module, then it is k−(T)−regularfor each k ≤ n, in particular, every n-(T)-regular R-module is 1-(T)-regular.

The following result gives a good motivaton for considering relativity in moduletheory.

Proposition 2.7: Let M be an R-module and T a sub module of M. If M is 1-locally (n-locally)-(T)-regular, then M/T is regular. The converse is true if M/T is locally projective.

Proof: Let N/T be a sub module of M/T and x∈N/T . Then there exists an Rhomomorphism $\alpha : M \rightarrow N$ such that $x - \alpha(x) \in T$. Hence α induces a mapping $\overline{\alpha} : M/T \rightarrow N/T$. This implies that $\overline{\alpha}(\overline{x}) = \overline{x}$ which means that M/T is regular. Conversly, let N be a sub module of M and x1, x2, ..., xn be a finite number of elements of N. Then there is an R-homomorphism s : M/T → N + T/T such that $s(\overline{x_i}) = \overline{x_i}$ for each i = 1, 2, ..., n. Local projectivity of M/T implies that there is an R-homomorphism$\overline{s}$: M/T → N such that $\pi \circ \overline{s}(\overline{x_i}) = s(\overline{x_i})$ for each i = 1, 2, ..., n, where π is the natural R-epimorphism of N onto N +T/T . Put $\pi = \overline{s}$ □$\pi : M \rightarrow N$. Then $\alpha(xi) - xi \in T$ for each i = 1, 2, ..., n and hence M is n − (T) − regular.

If M is n-(T)-regular R-module where T is a fully invariant sub module of M,then M is m-(T)-regular for each m n. Also, if M is n-(0)-regular R-module,then it is n-(T)-regular for each sub module T of M.

Recall that a sub module N of an R-module M is (T)-direct summand in M, ifthere exists a sub module K of M such that M = N + K and K ∩ N ⊆ T, where T is a sub module of M [1]. Let Q be the group of rational numbers and p be a prime number. Consider the two subgroups of Q, Qp ={a/b ∈Q : b is relatively prime to p } and Qp={a/pn ∈Q: n is non-negative integer } . Then, it is known that Qp +Qp = Q and Qp ∩ Qp = Z. Thus Qp (respectivilyQp) is (Z)-direct summand of Q while neither one is direct summand.

Proposition 2.8: Let M be an R-module and T a sub module of M.Then

(1) If M is n-(T)-regular, then every k-generated sub module of M is (T)-direct summand where $k \leq n$.

(2) If further, T is fully invariant in M, then every finitely generated sub moduleof M is (T)-direct summand.

Proof: (1) Let N $=\sum_{i=1}^{k} x_i$ R be k-generated sub module of M, for each $k \leq n$.By hypothesis, there exists an R-homomorphism s : M →N such thats(xi) − xi_T for each i = 1, 2, ..., k, and hence s(x) − x ∈T for each x in N. For each m ∈M, we have s(m) ∈N and s(s(m)−m) = s(s(m))− s(m) ∈N ∈T. This shows that M = N + s−1(T ∩ N), and it is easy to check that N ∩ s−1(T ∩N) ⊆ T. Thus N is(T)-direct summand.

(2) Let N be m-generated sub module of M. Without loss of generality, we canassume that m > n. As T fully invariant, then M is m−(T)−regular andhence by (1), N is (T)-direct summand.

Recall that a sub module N of an R-module M is (T)-direct summand in M, ifthere exists a sub module K of M such that M = N + K and K ∩N⊆T, whereT is a sub module of M [1]. Let Q be the group of rational numbers and p be aprime number. Consider the two subgroups of Q, Qp ={a/b ∈Q : b is relatively prime to p } and Qp={a/pn ∈Q: n is non-negative integer } . Then, it is known that Qp +Qp = Q and Qp ∩ Qp = Z. Thus Qp (respectivilyQp) is (Z)-direct summand of Q while neither one is direct summand.

Proposition 2.9: Let M be an R-module and T a sub module of M.Then

(1) If M is n-(T)-regular, then every k-generated sub module of M is (T)-direct summand where $k \leq n$.

(2) If further, T is fully invariant in M, then every finitely generated sub moduleof M is (T)-direct summand.

Proof: (1) Let N =$\sum_{i=1}^{k} x_i$ R be k-generated sub module of M, for each k ≤ n. By hypothesis, there exists an R-homomorphism s : M →N such that s(xi) − xi ∈T for each i = 1, 2, ..., k, and hence s(x) − x ∈T for each x in N. For each m ∈M, we have s(m) ∈N and s(s(m)−m) = s(s(m))−s(m) ∈N ∩T. This shows that M = N + s−1(T ∩ N), and it is easy to check that N ∩ s−1(T ∩ N) ⊆ T. Thus N is(T)-direct summand.

(2) Let N be m-generated sub module of M. Without loss of generality, we can assume that m > n. As T fully invariant, then M is m−(T)−regular and hence by (1), N is (T)-direct summand.

Corollary 2.10: Let M be an R-module and T asub module of M. If M is n-(T)- regular, then J(M) ⊆ T.

Proof: Let x ∈J(M). Then xR is small and (T)-direct summand of M. This implies that J(M) ⊆T.

Proposition 2.11: Let M be an R-module and T a fully invariant sub module of M. If M is n-(T)-regular and S is the endomorphism ring of M, then, as an S-module, M is n-(T)-regular.

Proof: We consider M a left S- module and hence (S −R)-bimodule. Let N be an S-sub module of M and x0∈N. Then there exists an R-homomorphism s : M → N such that s(x0) – x0∈T . We consider s is an element of S. Define ś : M → N byś(y) = s·y. It is clear that s

0 is an S-homomorphism and hence ś(x0)−x0∈T . This shows that N is 1-locally (and hence n-locally) (T)-split. Thus M is n -(T)-regular

S-module.

3. N-Locally Projective Modules Relative To A Sub module

Locally Split Homomorphisms Relative To A Sub module

Projectivity relative to a sub module had been studied in [1]. Let M be anR-module and T a sub module of M. M is called (T)-projective, if for each Repimorphism $\alpha : A \rightarrow B$ and R-homomorphism $\beta : M \rightarrow B$, there exists an R-homomorphism $\theta : M \rightarrow A$ such that $\alpha \circ \theta(x) - \beta(x) \in \beta(T)$ for each x in M.

In this section, we consider the local property of (T)-projective modules. We characterize these modules by means of locally homomorphisms relative to a sub module. First we introduce the following:

Definition 3.1: Let M be an R-module, T a sub module of M and n a positive integer. M is called n-locally projective relative to T (or simply n-locally (T)- projective), if for each R-epimorphism $\alpha : A \rightarrow B$ and R-homomorphism $\beta : M \rightarrow B$, then for any finite number of x1, x2, ..., xn_M, there exists an R-homomorphism $\sigma : M \rightarrow A$ such that $\alpha \circ\sigma (x_i) - \beta(x_i) \in \beta(T)$ for each i = 1, 2, ..., n.

It is clear that, every (T)-projective module is n-locally(T)-projective for each positive integer n and each sub module T. In particular, every projective module is locally projective module which introduced by Zimmermann in [11]. Also, it is clear that n-locally (T)-projective module is (T)-projective if it is finitely generated by n elements.

The Z-module Q is not 1-locally (Z)-projective, if not, let x_Q which is not in Z.Assume F is a free Z-module having a Z-epimorphism$\omega : F \rightarrow Q$, then there exists a Z-homomorphism $f : Q \rightarrow F$ such that $\omega \circ f(x) - x \in Z$. But HomZ(Q,Z) = 0and hence HomZ(Q, F) = 0. This implies that $x \in Z$ which contradicts the choiceof x. More generally, KR is not 1-locally (R)-projective where R is a domain and K is the field of quotients of R as R-module.

In the following , we give characterizations of n-locally (T)-projective modules in terms of n-locally (T)-split homeomorphisms.

Theorem 3.2: The following are equivalent for an R-module M and a sub module T of M

(1) M is n-locally (resp. 1-locally) (T)-projective,

(2) Every R-epimorphism into M (from any R-module) is n-locally(resp.1-locally) (T)-split,

(3) For any finite number of x1, x2, .., xn (resp.x) ∈M, there exist families $\{m_j\}_{j\in J} \subseteq M$ and $\{\varphi_j\}_{j\in J} \subseteq M^*$ such that for each i = 1, 2, ..., n

(a) φ_j (xi)(resp. φ_j (x)) ≠ 0 for only finitely many j ∈ J

(b) xi $-\sum_{j\in J} m_j \varphi_j$ (xi) (resp.x $-\sum_{j\in J} m_j \varphi_j$ (x)) ∈T .

Proof: We shall prove the n-locally case

(1) ⟹ (2) : Let A be any R-module and α : A → M be an R-epimorphism .Then there exists an R-homomorphism β : M → A such that α(β(xi) – xi ∈T for each i = 1, 2, ..., n. This shows that α is n-locally (T)-split.

(2) ⟹ (3) : Let $\{m_j\}_{j\in J}$ be a generated set of M, that is M $=\sum_{j\in J} m_j R$. Define f : $\oplus_{j\in J}$Rj → M where Rj = R for every j∈J, by f((rj)) $=\sum_{j\in J} m_j rj$. Clearly, f isan R-homomorphism. By (2), there is an R-homomorphism φ : M →$\oplus_{j\in J}$Rj such that f(φ(xi))–xi ∈T for each i = 1, 2, ..., n. Then for each j ∈ J, there is φj : M →Rj such that φ(m) = (φj (m)) for each m ∈ M, in particular, φ(xi) = (φj(xi)) for each i = 1, 2, ..., n. Thus xi $-\sum_{j\in J} m_j$φj (xi) = xi −f(φj (xi)) = xi − f(φj (xi)) ∈T . Let J0 = {j∈J : φj (xi) ≠ 0}. Then J0 is a finite subset of J and xi $-\sum_{j\in J} m_j$φj (xi) ∈Tfor each i = 1, 2, ..., n.

(3) ⟹ (1) : Let α : A → B be an R-epimorphism and β : M → B be an R-homomorphism. By (3), for each finite number of elements x1, x2, ..., xn of

M, there are families $\{m_j\}_{j\in J} \subseteq$ M and $\{\varphi_j\}_{j\in J} \subseteq$ M* such that φj (xi) ≠ 0 for finitely many j ∈J and xi−$\sum_{j\in J} m_j\varphi j$ (xi) ∈ T for each i = 1, 2, ..., n. Since β(mj) ∈ B, there exists aj∈ A such that α(aj) = β(mj) for each j ∈ J. Define σ : M → A by σ(m) =$\sum_{j\in J} a_j\varphi j$ (m) for each m ∈ M, in particular σ(xi) =$\sum_{j\in J} a_j\varphi j$ (xi) for each i = 1, 2, ..., n. Thus α∘ σ(xi)− β(xi) = α($\sum_{j\in J} a_j\varphi j$ (xi))− β(xi) =$\sum_{j\in J} \alpha(a_j)\varphi j$ (xi) − β(xi) =$\sum_{j\in J} \beta(m_j)\varphi j$ (xi) − β(xi) = β($\sum_{j\in J} m_j\varphi j$ (xi)) − xi) ∈ β(T).This shows that M is n-locally (T)-projective. We call the third statement of the above theorem, the dual basis lemma forn-locally(T)-projectiive modules. the original statements in the theorem are equivalent to the respective statements for fully invariant sub modules, more precisely, every n-locally(T)-projective module is k-locally(T)-projective for each k ≤ n. Also,if M is 1-locally(T)-projective module and T is fully invariant in M, then M is n-locally(T)-projective.

Corollary 3.3: The following statements are equivalent for an R- module M and a sub module T of M:

(1) M is n-locally(resp.1-locally)(T)-projective,

(2) For each free R-module F, each R-epimorphism β : F → M is n-locally(resp.1- locally)(T)-split.

Proof. (1) ⟹ (2) :follows from theorem (3.2)

(2) ⟹ (1) : Let A be any R-module and β : A → M be an R-epimorphism. Let$\{a_i\}_{i\in I}$ be a generated set of A and let F be a free R-module with basis $\{z_i\}_{i\in I}$. Define γ : F → A by γ(zi) = ai. Clearly is an R-epimorphism. By (2), β∘γ is n-locally (T)-split, that is for any finite number of x1, x2, .., xn∈ M, there exists an R homomorphism φ : M → F such that β∘γ (φ(xi))−xi∈ T for each i = 1, 2, ..., n. Put ς = γ ∘φ, then ς : M → A and satisfies

$\beta(\varsigma(xi)) - xi \in T$ for each i = 1, 2, ..., n. This shows that β is n-locally(T)-split and hence by theorem(3.2), M is n-locally(T)-projective.

Remark 3.4: (1) Let K be an R-module which is not (T)-projective for some sub module T of K (Q is not (Z)-projective Z-module), and M = K ⊕ H where H = ⊕R is a direct sum of countable number of copies of R.

Since every module is projective relative to itself, then H is (H)-projective and hence M is not (T⊕H)-projective, otherwise, by ([1],propositon(3.7))implies that K is (T)-proective. We claim that M is 1 − locally(T ⊕ H)−projective . Let $\{x_i\}_{i \in N}$be a basis for H. Then $T \cap \{x_i\}_{i \square N}$ is a generated set of M. For each j∈N, define fj : H → R by f(j)(xi)

$$= \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

fj can be extended (by linearity) to all H, therefore, if $x = \sum_{i=1}^{n} x_i r_i$, then fj(x) = rj . Again fj can be extended to an R-homomorphism gj : M → R by putting gj(t) = 0 for all t ∈T . Then {gj} ⊆M*. Let m ∈M. It is clear that gj(m) ≠ 0 for only finitely many j ∈N and m = t + x where t ∈T and x ∈H. Then we have $m - \sum_{j=1}^{n} x_j gj(m) = m - \sum_{j=1}^{n} x_j fj(m) = m - x \in T$. By dual basis lemma for 1-locally(T)-projective modules we have M is 1-locally(T)-projective and hence 1-locally(T ⊕ H)-projective.

(2) As an application of theorem(3.2), it is easy to see the following: If R is a commutative ring and Mi is 1-locally(Ti)-projective R-module,i = 1, 2. Then M1⊗ M2 is 1-locally(M1⊗ T2 + T1⊗ M2 + T1⊗ T2)-projective . In particular, tensor product of projective module M with 1-locally(T)-projective module is 1-locally(M ⊗ T)-projective.

(3) the following statement is obvious. It follows directly by the dual basis lemma for 1-locally(T)-projective modules. Let R be a commutative ring.

If M is 1-locally(T)-projective R-module and S is a multiplicative closed subset of R, then S−1M is 1-locally(S−1T)-projective S−1R-module, inparticular MP is 1-locally(TP)projective RP-module for each prime ideal P of R.

(4) The converse of (3) is not true in general, as we see in the following example. Let R be a Von Neumann regular ring which has no finitely generated maximal ideal and hence has no maximal ideal which is a direct summand. Let$\{P_\alpha\}_{\alpha\square\square}$ be the family of maximal ideals of R. Consider the R-module M = $\prod_{\alpha\in\Lambda}$(R/Pα). Then R_{P_α} is a field and hence M_{P_α} is 1-locally(T)- projective R_{P_α}-module for each sub module (T) of M. We claim that M* = 0. For, if f : R/Pα → R, then either f = 0 or f is R-monomorphism. If f is R-monomorphism, then R/Pα is isomorphic to an ideal Wα in R. As R regular, then Wα is a pure in R, and Pα = annR(Wα). Thus WαPα = Wα ∩ Pα = 0. Maximality of P_ implies that Pα+Wα = R and hence Pα is a direct summand of R, which contradicts the choice of R. Thus M* = 0. Let K be a proper sub module of M and m ∈M/K. By the above, MP is 1−locally(KP)−projective R P−module. If M is 1-locally(K)-projective R-module, then by the dual basis lemma for 1-locally(K)-projective modules we have m ∈K which is a contradiction.

(5) IfM is 1-locally (and hence n-locally)(T)-projective R-module, then J(M) ⊆MJ(R)+T(direct application of dual basis lemma for 1-locally(T)-projective modules).Further, if T a small sub module of M, then J(M) = MJ(R)+T

(6) The following result gives a motivation for studying n-locally(T)-projective modules: Let M be an R-module and T a sub module of M. Then M is 1- locally(resp. n-locally)(T)-projective R-module if and only if M/T is locally projective R-module.

Proof: Let $\bar{x}$ ∈M/T . then there exist a pair of dual basis $\{x_j,\varphi_j\}_{j\in J}$ on M such that φ_j (x) ≠0 for only finitely many j ∈J and x – $\sum_{j\in J} x_j \varphi_j(x)$ ∈T . For each j ∈J , define $\bar{\varphi}_j$: M/T → R/t(T) by $\bar{\varphi}_j$ ($\bar{m}$) = φ_j(m) + t(T) form ∈M where t(T) is the trace ideal of T. It is clear that $\bar{\varphi}_j$ is well-defined R/t(T)-homomorphism and $\bar{x}$ =$\sum_{j\in J} \bar{x}_j\bar{\varphi}_j(\bar{x})$. This shows that M/T is locally projective R/t(T)-module. As t(T) is two-sided ideal of R, then M/T is locally projective R-module. Conversely, let x ∈M. Then there a pair of dual basis $\{\bar{x}_j ,\bar{\varphi}_j\}_{j\in J}$ on M/T such that $\bar{\varphi}_j(\bar{x})$ ≠ 0 for only finitely many j ∈J and $\bar{x}$ =$\sum_{j\in J} \bar{x}_j\bar{\varphi}_j$(x). Thus x−$\sum_{j\in J} \bar{x}_j\bar{\varphi}_j \circ \pi(x)$ ∈T where π is the natural epimorphism. This shows that M is 1-locally(T)-projective.

It is well-known that, if M is a projective R-module, then M = Mt(T), and (M) =and (t(M)) and t(M) is a pure ideal of R , provided that R is a commutative ring, where t(M) =∑ α(M), the sum runs over all α ∈ M*. For 1-locally(T)-projective modules, we have the following.

Proposition 3.5: Let R be a commutative ring and M a 1-locally(T)-projective R-module. Then

(1) M = Mt(M) = T

(2) annR(M) = annR(t(M)) ∩annR(T)

(3) t(M) = (t(M))2 + t(T)

Proof:(1) By theorem(3.2), for each x ∈M, x =$\sum_{i\in I} x_i$fi(x)+vi where fi∈M*, xi ∈M and vi∈T . Therefore x ∈Mt(M) + T and hence M ⊆ Mt(M) + T. Thus M = Mt(M) + T.

(2) Let w ∈t(M).Then w =$\sum_{i=1}^{n} f_i$(ai) whereai∈M. Let r ∈annR(M). Then wr =$\sum_{i=1}^{n} f_i$(air) = 0, hence r ∈annR(t(M)) . Thus annR(M) ⊆ annR(t(M)∩

annR(T). Let s_annR(t(M)) ∩annR(T). Then by (1), we have Ms = Mt(M)s + Ts = 0. Thus annR(M) = annR(t(M)) ∩annR(T).

(3) Let w_t(M). Then w $=\sum_{i\in I} h_i$(mi) where mi∈M, hi∈M*. For each I ∈I, mi $=\sum_{j\square J} y_j\, \varphi_j$(mi)+vi where yj∈M, φ_j ∈M* andvi∈T . w $=\sum_{i\square I} h_i(\sum_{j\square J} y_j\, \varphi_j$(mi) + vi) $=\sum_{i\square I}\sum_{j\square J} h_i\,(y_j)\, \varphi_j$(mi)+$\sum_{i\square I} h_i$(vi). Thus w ∈(t(M))2+t(M)\t(T) and hence t(M) = (t(M))2 + t(T).

Corollary 3.6: Let R be a commutatvie ring, M an R-module and T a sub moduleof M. If M is 1-locally(T)-projective, then

(1) Mt(M) is (T)-direct summand (and hence(T)-pure) in M.

(2) t(M) is (t(T))-pure ideal of R.

Theorem 3.7: The following statements are equivalent for an R-module M and a sub module T of M:

(1) M is n-locally(T)-projective,

(2) For each k-generated sub module M0 of M where k ≤ n, there exist a finitely generated free R-module F and R-homomorphisms f : M → F and g : F → M such that g(f(x)) – x ∈T for each x ∈M0.

Proof: (1) ⟹ (2) : Let Qbe a free R-module having an R-epimorphism h : Q → M. Then h is n-locally(T)-split. Thus we can find an R-homomorphism q : M → Q such that h(q(x)) − x ∈T for all x ∈M0. As q(M0) is a finitely generated sub module of Q, there exists a finite subset {u1, u2, ..., uk} of the free basis of Q such that q(M0) is contained in a finitely generated free sub module F = u1R + u2 + ... + ukR of Q. Since F is a direct summand of Q, then let ρ : Q → F be the natural projection of Q onto F. Put f = ρ ∘ q : M → F and g = h|F : F → M. Then clearly g(f(x)) – x ∈T for each x ∈M0.

(2) ⟹ (1) : Consider a finite number of x1, x2, ..., xn_M and let N be the

sub module of M generated by these elements. By the hypothesis, there exist a finitely generated free R-module F and R-homomorphisms f : M → F, g : F → M such that g(f(x)) – x ∈T for each x_N,in particular g(f(xi)) − xi ∈T for each i = 1, 2, ..., n. Let {u1, u2, .., uk} be a free basis for F. For each j = 1, 2, .., k we define an R-homomorphism φ_j : M → R by f(m) $=\sum_{j=1}^{k} u_j\varphi_j(m)$ for each m ∈M. Let yj = g(uj) for each j = 1, 2, ..., k. Then for each i = 1, 2, ..., n we have$\sum_{j=1}^{k} y_j\varphi_j(x_i)$ − xi $=\sum_{j=1}^{k} g(u_j)\varphi_j(xi)$ − xi = $g(\sum_{j=1}^{k} u_j\varphi_j(xi))$ − xi = g(f(xi)) − xi∈T . Theorem(3.2) implies that M is n-locally(T)-projective. _

Corollary 3.8: If M is n-locally(T)projective R-module. then for every k-generated sub module N of M where k ≤ n, there exists s ∈EndR(M)suchthat s(x) – x ∈T for each x ∈N.

The last corollary suggests a weak concept of n-locally(T)-split sub modules. LetN be a sub module ofM. N is called weak n-locally(T)-split if for each finite number of x1, x2, ..., xn∈N,there exists an R-endomorphism s of M such that s(xi) − xi∈T for each i=1,2,...,n. It is clear that n-locally(T)-split sub modules are weak n-locally(T)-split. The converse is not true. Thus, in n-(T)-regular modules, every sub module is weak n-locally(T)-split, while in n-locally(T)-projective modules, every k-generated sub module is weak n-locally(T)-split where k ≤ n.

We have mentioned before that every k-generated n-locally(T)-projective module where k ≤ n is (T)-projective, for countably generated modules we have the following:

Theorem 3.9: Let M be an R-module and T a fully invariant sub module of M. If M is countably generated n-locally(T)-projective module, then it is (T)-projective.

Proof:Let {x1, x2, x3, ...} be a countably generated set of M. Let M1 = x1R. Then theorem (3.7) implies that there are a finitely generated free R-module F1 and R-homomorphisms
f1 : M → F1, g1 : F1 → M such that g1(f1(x)) – x ∈T for each
x ∈M1. Let M2 = g1(F1) + x2R. Since M2 is finitely generated, again by theorem (3.7), there exist a finitely generated free R-module F2 and R-homomorphismsf2 : M → F2, g2 : F2 → M such that g2(f2(x)) – x ∈T for each x ∈M2. Observethat g1(F1) ⊆ g2(F2) + T and x2∈g2(F2) + T. In this manner, for each n > 1, wecan find a finitely generated free R-module Fn and R-homomorphismsfn : M → Fn, gn : Fn → M such that gn(fn(x)) – x ∈T for each x ∈Mn = gn−1(Fn−1) + xnR.
This is equivalent to saying that gn(fn(gn−1(y)))−gn−1(y) ∈T for each y ∈Fn−1 and gn(fn(xn)) − xn∈T and hence gn−1(Fn−1) ⊆ gn(Fn) + T and xn∈gn(Fn) + T. Thus we have an ascending chain g1(F1) ⊂ g2(F2) + T ⊂g3(F3) + T ⊂ ... of sub modules of M whose union is equal to M. Let sn = gn∘ fn : M →gn(Fn) for each n . Then sn∈EndR(M) satisfying that sn _ gn−1(m) − gn−1(m) ∈T and hence sn∘ sn−1(m)−sn−1(m) ∈T for each m_M and n > 1. Thus sn∘ gr(m) – m ∈T and sn∘ sr(m) − sr(m) ∈T for each m ∈M, whenever r < n, because gr(Fr) + T ⊂gn−1(Fn−1) + T and so sn(gr(y)) − gr(y) ∈T for every y ∈Fr. We shall convene that for any two R-homomorphisms α and β, α = β modulo T means α(x)− β(x) ∈T for
each x in their common domain. Let F = $\oplus_n$Fn. Then F is a countably generated free module. Define g : F → M by g((wn)n) =$\sum g_n$(wn), n=1,2,3,... . Thus g(F) =$\sum g_n$(Fn) + T = M and hence g is an R-epimorphism. We claim that g is (T)-split R-homomorphism, that is, there exists an R-homomorphism f : M → F such that g ∘f(x)−x ∈T for all x ∈M.

Now, let qn : Fn → F be the canonical injection for each n. Then g ∘qn = gn. We shall construct an R-homomorphism hn : Fn → F such that g ∘ hn = gn modulo T and hn∘ fn∘gn−1 = hn+1∘ fn+1∘ gn−1 modulo T if n > 1 by induction on n. Let h1 = q1. Then g ∘ h1 = g1. Suppose n > 1 and thereis given an R-homomorphism hn : Fn → F such that g ∘ hn = gn modulo T andhn∘fn∘gn−1 = hn+1∘fn+1∘gn−1 modulo T. We define hn+1 = (hn∘fn+gn+2∘fn+2∘ (1−sn)) ∘ gn+1. Then we have g ∘hn+1 = (g ∘hn∘fn+gn+2∘ fn+2∘ (1−sn)) ∘ gn+1= (gn∘ fn +gn+2∘ fn+2∘ (1−sn)) ∘ gn+1 modulo T = (sn +sn+2∘ (1−sn)) ∘ gn+1modulo T = (sn + sn+1 − sn+2∘sn) ∘ gn+1 modulo T = sn+2∘ gn+1 modulo T = gn+1 modulo T. On the other hand, we have hn+1∘ fn+1∘ gn−1 = (hn∘ fn + gn+2∘ fn+2∘ (1 −sn)) ∘ gn+1∘ fn+1∘ gn−1 = (hn∘ fn + gn+2∘ fn+2∘ (1 − sn)) ∘ gn modulo T = hn∘ fn∘ gn−1 + gn+2∘ fn+1∘gn−1 − gn+2∘ fn+2∘ gn−1 modulo T= hn∘ fn∘ gn−1modulo T. Thus we get a desired sequence of R-homomorphismshn. Let x∈M. Then there exists n > 0 such that x ∈gn+1(Fn−1) + T, that is, x = gn−1(y)+t for some t ∈T and y ∈Fn−1. Then we have hn(fn(x)) = hn(fn(gn−1(y)+ t)) = hn+1(fn+1(gn−1(y) + t)) + t1 = hn+1(fn+1(x)) + t1 for some t1∈T . Moreover, since x ∈gn(Fn) + T, in this case, by replacing n by n+1 we should have hn+1(fn+1(x)) = hn+2(fn+2(x)) + t1. Continuing in this way, we confirm that hn(fn(x)) = hm(fm(x)) + t1 for every m >n. This shows that hn(fn(x)) is independent of the choice of n so for as x in gn−1(Fn−1). Define f(x) = hn(fn(x)) for each x ∈M, we have an R-homomorphism f : M → F, which satisfies g(f(x)) − x = gn(fn(x)) – x ∈T (since x ∈gn−1(Fn−1)). Finally, the R-module F is projective and hence F is (f(T)) − projective. Then f(m) ∈F . By dual basis lemma for (f(T))-projective modules ([1],theorem(3.8)) there exist two families $\{w_j\}_{j \in J}$ ⊆F and $\{\varphi_j\}_{j\in J} \subseteq F^*$ such that $\varphi_j(f(m)) \neq 0$ for only finitely many $j \in J$ and $f(m) - \sum_j w_j \varphi_j(f(m)) \in f(T)$. But $m = g(f(m)) + v$ for some $v \in T$. Thus $m = g(\sum_j w_j \varphi_j(f(m)) + f(t) - v = \sum_j g(w_j)\varphi_j \circ f(m) + g f(t) - v = \sum_j g(w_j)(\varphi_j \circ f)(m) + t1$ for some $t, t1 \in T$. Thus the two families $\{g(w_j)\}_{j\in J}$ and $\{\varphi_j \circ f\}_{j\in J}$ satisfy the dual basis lemma for (T)-projective modules and hence M is (T)-projective.

It is well-known that, every projective module is isomorphic to a direct summanda free module. In [11], it was proved that, every locally projective R-module is a pure sub module of a direct product of copies of R. For n-locally (T)-projective modules we have the following.

Proposition 3.10: Every n-locally (T)-projective R-module is isomorphic to a (t(T)I)-pure sub module of RI

Proof: Let M be n-locally (T)-projective R-module and denote M* as a family$(f_i)_{i\in I}$. Define $\theta : M \rightarrow RI$ by $\theta(m) = (f_i(m))_{i\in I}$ for $m \in M$. Clearly θ is an R-homomorphism and M is isomorphic to θ(M). We claim that θ(M) is (t(T)I) –pure sub module of RI . Consider a system of equations $\theta(mk) = (f_i(m_k))_{i\in I} = \sum_{l\in L} r_l s_{lk}$ where $rl = (r_{li})_{i\in I} \in RI$, $slk \in R$, L is a finite set and $k \in K$ (finite set). Theorem (3.2) implies that there exist a finite subset $J \in I$ and a family$\{x_j\}_{j\in J} \subseteq M$ such that $mk = \sum_{j\in J} x_j fj(mk) + tk$ where $tk \in T$ for all $k \in K$. $\theta(mk) = (f_i(\sum_{j\in J} x_j f_j(m_k) + t_k)_{i\in I} = (f_i \left(\sum_{j\in J} f_i(x_j) f_j(m_k) + f_i(t_k)\right)_{i\in I} = \sum_{j\in J} f_i(x_j) \sum_{l\in L} r_{lj} s_{lk} + (f_i(t_k))_{i\in I} = \sum_l f_i(\sum_j x_j r_{lj}) s_{lk} + (f_i(t_k))_{i\in I}$ for all i_I, k_K. But $(f_i(t_k))_{i\in I} \in t(T)I$.

Then $\theta(mk)-\sum_{l\square L}(x_j r_{lj})s_{lk} \in t(T)I$, but $\theta(xjrlj) \in \theta(M)$. This shows that $\theta(M)$ is $(t(T)I)$-pure sub module of RI .

Corollary 3.11: Every n-locally (T)-projective R-module is isomorphic to a (t(T)I)-pure sub module of direct product of free modules.

Proposition 3.12:Let M be n-locally (T)-projective R-module and N a sub moduleof M. If N+T is (T)-pure in M, then N is n-locally ($T \cap N$)-projective and n-locally ($T \cap N$)-split.

Proof: Let x1, x2, ..., xn$\in$N. Then by theorem(3.2), there are two families $\{m_j\}_{j\square J} \square$ M and $\{\varphi_j\}_{j\square J} \square$ M* such that for each i = 1, 2, ..., n, $\varphi_j(xi) \neq 0$ for only finitely many j $\in$J and $xi - ti = \sum m_j\varphi_j(x_i)$. (T)-purity of N + T in M implies that there exists uj$\in$N and vj$\in$T such that $xi - ti - ((\sum (u_j+v_j)\varphi_j(x_i) \in T$. Thus $xi - uj(\varphi_j|N)(xi) \in T \cap N$. This implies that N is n-locally ($T \cap N$)-projective. Define

$s : M \to N$ by $s(m) = \sum u_j(\varphi_j|N)(m)$ for each m $\in$M. Clearly, $s(xi) - xi \in T \cap N$ for each i = 1, 2, ..., n and hence N is n-locally ($T \cap N$)-split.

Corollary 3.13: Let M be a locally projective R-module and N a pure sub moduleof M. Then N is locally projective and locally split.

Proposition 3.14: Let M be an R-module and T a fully invariant sub module of M. If M is n-locally (T)-projective R-module and S is the endomorphism ring of M, then ,as an S-module, M is n-locally (T)-projective.

Proof: Let A be a left S-module and $\alpha : A \to M$ an S-epimorphism. Let $x0 \in \alpha(A)$. Then α is an R-homomorphism. Thus n-local (T)-projectivity of M implies that α is 1-locally (T)-split, that is, there is an R-homomorphism $\beta : M \to A$ such that $\alpha(\beta(x0)) - x0 \in T$. Let y $\in$M. Then the mapping $\overline{\beta} : M \to A$ defined by $\overline{\beta}(y) = \beta.y$ is an S-homomorphism and $\alpha(\overline{\beta}(x0)) - x0 =$

$\alpha(\beta(x0)) - x0 \in T$. Then theorem (3.2) and proposition (2.2) imply that M is n-locally (T)-projective S-module.

4. MORE REGULARITY RELATIVE TO A SUB MODULE

In this section, we invest locally (T)-split homomorphisms in studying Zelmanowitzregularand Field house-regular modules relative to a submodul.

Definition 4.1: Let M be an R-module and T a sub module of M. Then

(1) M is called Field house-regular relative to T (Simply,Field house (T)-regular), if each sub module of M is (T)-pure.

(2) M is called Zelmanowitz-regular relative to T(Simply,Zelmanowitz(T)-regular), if for each m $\in M$ there is $\alpha \in M^*$ such that $m - m\alpha(m) \in T$.

It is clear that, an R-module M is Zelmanowitz (Field house)-regular if and onlyif, it is Zelmanowitz (Field house)(0)-regular. If an R-module M is Zelmanowitz(Field house)(T1)-regular, then M is Zelmanowitz (Field house)(T2)-regular for each sub module T2 of M containing T1. Then every Zelmanowitz (Field house) regular module is Zelmanowitz (Field house)(T)-regular for each sub module T of M.

Remark 4.2: (1) Every sub module N of Zelmanowitz (Field house)(T)-regular module M, is Zelmanowitz (Field house)($T \cap N$)-regular.

(2) For each positive integer n, the Z-module Zn is not Zelmanowitz (T)-regular for each proper sub module T of Z.If not, let $\bar{a} \in Zn/T$. Then there is Zhomomorphism $\alpha : Zn \to Z$ such that$\bar{a} - \bar{a}\,\alpha(\bar{a}) \in T$. But HomZ(Zn,Z) = 0, this implies that $\bar{a} \in T$ which contradicts the choice of $\bar{a}$. In a similar manner we can see that Z-modules Q and Zp1 is not Zelmanowitz (T)- regular for each proper sub module T of Q and Zp1respectivily.

(3) If $\alpha : M \to N$ is an R-epimorphism and M is Field house (T)-regular, then N is Field house ($\alpha(T)$)-regular.

(4) Every n-(T)-regular module is Field house(T)-regular,(see proposition(2.5)).

(5) Let M be an R-module and T a sub module of M. Then M is Field house (T)-regular R-module if and only if M/T is Field house regular R-module.

Proof: Let$\bar{n}_j = \sum_{i=1}^{n} \bar{x}_i rij$ where $\bar{n}_j \in N/T$, $\bar{x}_i \in M/T$, rij∈R and j = 1, 2, ...,m. Then nj $-\sum_{i=1}^{n} \bar{x}_i rij \in T$ and hence (nj + tj) = $\sum_{i=1}^{n} \bar{x}_i rij$ where tj∈T ⊆ N, j = 1, 2, ...,m. There exist $\acute{x}_i \in N$ such that (nj + tj) $-\sum_{i=1}^{n} \acute{x}_i rij \in T$ and hence $\bar{n}_j = \sum_{i=1}^{n} \acute{x}_i rij$. Conversely, let N be a sub module of M and nj $=\sum_{i=1}^{n} x_i rij$ where nj∈N, xi∈M and rij∈R, j = 1, 2, ...,m. As N + T/T is pure in M/T , there exist$\acute{x}_i \in N + T/T$ such that $\bar{n}_j = \sum_{i=1}^{n} \acute{x}_i rij$ and hence nj $-\sum_{i=1}^{n} \acute{x}_i rij \in T$. Thus M is Field house (T)-regular.

In the following, we characterize Field house (T)-regular modules over a commutative
rings.

Proposition 4.3: Let M be a module over a commtative ring R and T a suboduleof M. Then the following are equivalent:

(1) R/(T : x) is regular ring for each non-zero element x ∈M,

(2) For each x ∈M and r ∈R, there exists s ∈R such that rx – rsrx∈T ,

(3) M is Field house (T)-regular.

Proof: (1) ⟹ (2) : Let x ∈M and r ∈R. Since R/(T : x) is regular, then there exists $\bar{s}$ ∈R/(T : x) such that r = rsr and hence rx – rsrx∈T .

(2) ⟹ (3) :Let N be a sub module of M and A an ideal of R. Let x ∈N ∩ MA. Then x $=\sum_{i=1}^{n} m_i ai$ where mi∈M and ai∈A. By (2), for each i = 1, 2, ..., n there is si∈R such that miai − miaisiai∈T . Put ei = siai and e = $\prod_{i=1}^{n}(1-e_i)$, note that e ∈A and miai − miaiei = wi∈T for each i = 1, 2, ..., n and miei − mie_i^2 ∈T . It is easy to check that for each j, mjeiej− mjei = uj∈T . Now xe

$=\sum_{i=1}^{n} m_i aie$ $=\sum_{i=1}^{n} m_i aieie$ $+\sum_{i=1}^{n} w_i e$ $=\sum_{i=1}^{n} m_i aiei$ $+\sum_{i=1}^{n} u_i ai$ $+\sum_{i=1}^{n} w_i e$ $=\sum_{i=1}^{n} m_i ai$ $+ \sum_{i=1}^{n} u_i ai +\sum_{i=1}^{n} w_i(e - ei) = x + v$ where $v =\sum_{i=1}^{n} u_i ai + wi(e - ei) \in T$, thus $x \in NA + T \cap (MA \cap N)$. This shows that N is (T)-pure in M and hence M is

Field house(T)-regular.

(3) $\Longrightarrow$ (1): Let $x \in M$ and $\bar{r} \in R/(T : x)$ and let P be the sub module generated byy = xr. Then P is (T)-pure in M, so there is $z \in P$ such that $y - rz \in P \cap T$, so there is $r \in R$ such that z = trx and hence $\bar{r} = \overline{rtr}$.

Corollary 4.4: Let M be a module over a commutative ring R and T a sub moduleof M. Then

(1) If R/(T : M) is a regular ring, then M is Field house (T)-regular.

(2) If M is a finitely generated Field house (T)-regular R-module, then R/(T : M) is regular ring.

Proof: (1) It is easy to see that R/(T : x) is epimorphic image of R/(T : M) for each $x \in M$ and hence proposition (4.3) implies that M is Field house (T)-regular.

(2) Let {x1, x2, ..., xn} be generated set of M and A = (T : M),Ai = (T : xi) for each i = 1, 2, ..., n. Then $A = \bigcap_{i=1}^{n} A_i$. Define $\alpha : R/A \rightarrow \bigoplus_{i=1}^{n} R/Ai$ by $\alpha(r + A) = (r + A1, r + A2, ..., r + An)$ for $r \in R$. It is easy to see that α is Rmonomorphismand hence R/A is isomorphic to a subring W of $\bigoplus_{i=1}^{n} R/Ai$ where $W = \{(r+A1, r+A2, ..., r+An)|r \in R\}$. Note that $\bigoplus_{i=1}^{n} R/Ai$ is regular. We finish if we prove that W is regular. Let y = (r+A1, r+A2, ..., r+An) $\in W$. Then for each i, there is $ti \in R$ such that r +Ai = rtir +Ai and hence r $-rtir \in (T : xi)$. Put $1-tr =\prod_{i=1}^{n}(1-t_i r)$. Then for each i, $r(1-tr) = r(\prod_{i=1}^{n}(1-t_i r))=\prod_{i=1}^{n}(1-t_i r) \in (T : xi)$, this shows that r + Ai = rtr + Ai. If we put u = (t + A1, t + A2, ..., t + An), then y = yuy.

We have proved in section one that, if M is n-(T)-regular R-module, then J(M) ⊆ T and hence J(R) ⊆ (T : M). First we note that, if P is a (T)-pure sub module ofM and A a right ideal of R, then P ⊆ MA if and only if P = PA+(T∩P).

Lemma 4.5: If P is a finitely generated (T)-pure sub module in M such that P ⊆MA where A is a right ideal of R contained in J(R), then P ⊆ T.

Proof:By the above we have P = PA+(T ∩ P). Then Nagayama's lemma implies that PA is a small in P and hence P = T ∩P and thus P _ T.

Proposition 4.6: If M is a Field house (T)-regular R-module, then MJ(R) ⊆ T.

Proof: Let P be a finitely generated sub moduleof MJ(R). Since M is Field house (T)-regular, then P is (T)-pure in M and hence by lemma(4.5), P ⊆ T. This shows that MJ(R) ⊆ T.

Corollary 4.7: If M is a Field house (T)-regular R-module such that J(M) ⊂MJ(R) + T, then J(M) ⊂ T. In particular, J(M) ⊂T for every Field house (T)-regular 1-locally (T)-projective module.

Theorem 4.8: The following are equivalent for an R-module M and a sub moduleT of M:

(1) M is Zelmanowitz (T)-regular,

(2) Every R-homomorphism into M (form any R-module) is 1-locally(T)-split,

(3) Every R-homomorphism from R into M is 1-locally (T)-split.

Proof. (1) ⟹ (2) : Let α : A → M be an R-homomorphism and x ∈ α(A). Then x = α(z) for some z ∈A. By (1), there exists an R-homomorphism β : M → R such that x − xβ(x) ∈T . Define q : M →A by q(m) = zβ(m) for m

∈M. Then α(q(x)) − x = α(z)β(x)− x = xβ(x) – x ∈ T. This shows that α is 1−locally(T)−split

(2) ⟹ (3) : Trivial.

(3) ⟹ (1) : Let x_M. Define the well-defined R-homomorphism g : R → M by g(r) = mr for r ∈R. Then by (3), g is 1-locally(T)-split and hence there is an R-homomorphism q : M → R such that g(q(x)) – x∈T , that is xq(x) – x ∈T which implies that M is Zelmanowitz (T)-regular.

In the following, we see that the three types of regularity relative to a sub moduleare equivalent under 1-locally (T)-projective modules.

Theorem 4.9:The following statements are equivalent for an R-module M and a sub module T of M.

(1) M is Zelmanowitz (T)-regular,

(2) M is 1-locally(T)-projective and 1-(T)-regular,

(3) M is 1-locally(T)-projective and Field house(T)-regular.

Proof: (1) ⟹ (2) : Follows from the fact that every R-epimorphism(and every R-monomorphism) is 1-locally(T)-split.

(2) ⟹ (3) : Follows from examples andremarks(4.2)(4).

(3)⟹(1) : Let h : Q → M be an R-homomorphism. Since h(Q) + T is (T)-pure in M, then by proposition(3.12), h(Q) is 1-locally(T)-projective and 1-locally(T)-split. By regarding h as a map onto h(Q), we have an R-epimorphismh̀ : Q → h(Q). 1-local(T)-projectivity of h(Q) implies that h̀ is 1-locally(T)-split, theorem(3.2). Thus proposition(2.2) implies that h is 1-locally(T)-split. Therefore theorem(4.8) assert that M is Zelmanowitz(T)-regular.

Corollary 4.10: Let M be Zelmanowitz(T)-regular R-module where T is a fully invariant sub module of M and S the endomorphism ring of M. Then, as an SmoduleM is Zelmanowitz (T)-regular.

Proof: By theorem(4.9), M is 1-locally(T)-projective and 1-locally(T)-regular Rmodule. Hence proposition (3.14) and proposition (2.9) imply thatM is 1-locally(T)-projective S-module and 1-(T)-regular S-module. Again theorem(4.9) implies thatM is Zelmanowitz (T)-regular S-module.

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