Iraqi Journal of Statistical Sciences (25) 2013

The 6th Scientific Conference of the College of Computer Sciences & Mathematics

pp [14-20]

The Zero Divisor Graphs of a finite Certain Rings

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Abstract

For each commutative ring we associate a graph $\Gamma(R)$. In this paper we consider the zero divisor graphs of certain finite rings, and we characterize the complete bipartite zero divisor graph of finite commutative rings.

الملخص

لكل حلقة ابدالية R ، نمثل البيان $\Gamma(R)$. في هذا البحث درسنا بيانات قواسم الصفر للحلقات المنتهية، وكذلك اعطينا تمييزا لبيانات قواسم الصفر الثنائي التجزئة التام للحلقات المنتهية.

Keywords: zero divisor graph, regular ring, bipartite graph, diameter of the graph.

1. Introduction.

Throughout this paper , any ring R is a commutative with identity. We write $Z(R)^*$ the set of all non zero zero- divisor of a ring R. Recall from [1] that the zero divisor graph of R, denoted by $\Gamma(R)$, is the graph whose vertex set of all nonzero zero-divisors of R and distinct vertices x,y are joined by an edge in this graph if and only if xy=0.

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Received Date 24/9/2013 ———— Accept Date 21/11/2013

The notion of the zero divisor graph $\Gamma(R)$ was firstly introduced by I. Beck [3] , where his motive was coloring of the graph. This work had been extended to commutative semi-group by De Mayer, Mckanzie and Schneider [5]. The zero divisor graphs also studied by Anderson and Livinyston [1]. They found that $\Gamma(R)$ is always connected.

We recall that a graph G is said to be a bipartite if it's vertices can be divided into two disjoint sets U and V such that every edge connects a vertex in U to one in V; that is U and V are each independent sets. A graph G is said to be complete bipartite if every vertex of U is connected to every vertex of V. If $|U|=m$ and $|V|=n$, the complete bipartite graph will be denoted by $K_{m,n}$. Note that $K_{1,n}$ is called a star graph.

2. The Diameter of $\Gamma(R)$.

In this section we consider $\Gamma(R)$ with diameter less than or equal to2.Recall that in a graph G, the distance between two vertices u and v, d(u,v) is the length of the shortest path joining u and v. The diameter of G, denoted by dim(G), is the maximum distance among all pairs of vertices in G.

We start this section with the following result.

Theorem 2.1: Let R be a finite ring with $\dim(\Gamma(R)) \leq 2$. Then R is a local ring or $R \cong F_1 x F_2$, where F_1 and F_2 are fields.

Proof : Since R is a finite ring, then $R \cong R_1 x R_2 x \ldots x R_n$, where R_i are local ring for all $i \geq 1$.

If $n \geq 3$, then $Z_1=(0,1,1,\ldots,1)$ is only adjacent with $(1,0,0,\ldots,0)$, and $Z_2=(1,0,1,\ldots,1)$ is only adjacent with $(0,1,0,\ldots,0)$. Since Z_1 and Z_2 do not have a common annihilator, $\dim(\Gamma(R))=3$ which is a contradiction, therefore n=2 or 1.

Assume that $R\cong R_1xR_2$, if R_1 and R_2 are local rings but not a field. Let $z\in Z(R_1)^*$, $w\in Z(R_2)^*$. Consider the zero –divisor $z_1=(z,1)$ and $w_1=(1,w)$. The shortest path between z_1 and w_1 must be of length 3, and hence $\dim(\Gamma(R))=3$ which is a contradiction.

If R_1 is a local ring but not a field and R_2 is a field, let $x_1,x_2\in Z(R_1)^*$, such that $x_1x_2=0$. Consider the zero-divisor $z_1=(x_1,1)$ and $z_2=(1,0)$. The shortest path between z_1 and z_2 must be length 3, and hence $\dim(\Gamma(R))=3$ which is a contradiction. So that R is local or $R\cong F_1xF_2$. ■

Recall that , a ring R is said to be regular, if for every $a\in R$, there exists $b\in R$ such that a=aba. Clearly $(ab)^2$ =abab=ab. Therefore ab is an idempotent element in R.

Definition 2.2: We call a ring R, satisfies a property (*) if every orthogonal idempotent elements $e_1, e_2, \ldots,e_n$ we have $e_1+e_2+\ldots e_n=1$ for all positive integers $n\geq2$.

The following result establish the relation between regular ring R and the $\dim(\Gamma(R))$.

Theorem 2.3: Let R be a finite regular ring satisfying condition (*), then $\dim(\Gamma(R))\leq2$.

Proof : Let x,y be a distinct zero divisor elements in R. If xy=0, then d(x,y)=1. Suppose that $xy\neq0$, since R is regular, there exists $0\neq e=e^2$, $0\neq f=f^2\in R$ such that x=xe and y=yf, this implies that x(1-e)=0 and y(1-f)=0. If ef=0 ,then xy=xeyf=xyef=0 which is a contradiction. If $ef\neq0$, since R satisfying condition (*), then $(1-e)(1-f) \neq 0$, so that x(1-e)(1-

f)=0 and y(1-e)(1-f)=0. This means that the path between x,y is x— (1-e)(1-f) — y ,hence d(x,y)=2. Therefore dim(Γ(R))≤2. ■

The following example shows that the condition " regular ring, R satisfies a condition (*) or R finite ring" in Theorem 2.3 is not superfluous

Example 1:

1- $R=Z_{30}$ is a finite regular ring but is not satisfying condition (*), then dim(Γ(R))=3.

2- If $R=Z_{12}$ or $R=Z_{18}$, then R is a finite ring and not regular which satisfy the condition (*) and hence dim(Γ(R))=3.

3- Let $R=Z_2xZ$, then R is a infinite ring with Γ(R) is complete bipartite graph and R not regular. ■

Before giving the main result of this section, the following theorem which are due to Axtell, Stickles and Trampbachls [2] is needed.

Lemma 2.4: Let R be a finite ring. Then the following are equivalent:

1- Z(R) is an ideal;

2- Z(R) is a maximal ideal;

3- R is local;

4- Every $x\in Z(R)$ is nilpotent;

5- There exists $b\in Z(R)$ such that bZ(R)=0.

Theorem 2.5: Let R be a finite ring , then

1- dim(Γ(R)) =0 iff $R\cong Z_4$ or $Z_2[X]/(X^2)$.

2- dim(Γ(R)) =1 iff $R\cong Z_2xZ_2$ or R is local ring with $(Z(R))^2=0$.

3- dim(Γ(R)) =2 iff $R\cong F_1xF_2$ with $R\ncong Z_2xZ_2$ or local with $(Z(R))^2 \neq 0$, where F_1 and F_2 are field.

4- For all cases except (1), (2) and (3), dim(Γ(R)) =3.

<u>Proof :</u> (1) Since $\Gamma(R)$ connected and $\dim(\Gamma(R)) = 0$, then $\Gamma(R)$ has exactly one vertex so that by [4] $R \cong Z_4$ or $Z_2[X]/(X^2)$

(2) Let $\dim(\Gamma(R)) = 1$, then $\Gamma(R)$ is complete graph and therefore $R \cong Z_2 x Z_2$ or $xy=0$ for all $x,y \in Z(R)$[1]. If $R \cong Z_2 x Z_2$ we are done . If $xy=0$ for all $x,y \in Z(R)$, then by Lemma 2.4 R local and hence if $(Z(R))^2 \neq 0$, then there exists $a,b \in Z(R)$ such that $ab \neq 0$ which is a contradiction . Therefore $(Z(R))^2 = 0$.

(3) If $\dim(\Gamma(R)) = 2$, then by Theorem 2.3 R is local ring or $R \cong F_1 x F_2$. If $R \cong F_1 x F_2$, we are done. If R local, then by Lemma 2.4 there exists $b \in Z(R)$ such that $bZ(R)=0$,so that $\dim(\Gamma(R)) \leq 2$, if $(Z(R))^2 = 0$, then $\dim(\Gamma(R)) = 1$ which is a contradiction , therefore $(Z(R))^2 \neq 0$

(4) Since $\dim(\Gamma(R)) \leq 3$ for any ring [1, Theorem 2.3]. ■

3. Complete Bipartite Zero-Divisor Graph.

In this section we investigate a complete bipartite zero divisor graph $(\Gamma(R)$.

First we state the following result of [2]

<u>Lemma 3.1:</u> Let R a finite ring such that $\Gamma(R) = K_{1,n}$ with center a. Then the following are equivalent:

1- 1- Z(R) is an ideal;
2- $a^2 = 0$;
3- $R \cong Z_4, Z_8, Z_9, Z_2[X]/(X^2), Z_2[X]/(X^3), Z_3[X]/(X^2)$ or $Z_4[X]/(2X, X^2-2)$.

<u>Proposition 3.2:</u> Let R be a finite ring with $\Gamma(R)$ is complete bipartite, then $R\cong F_1xF_2$ or A, where F_1 and F_2 are fields and $A\cong Z_4$, Z_8, Z_9, $Z_2[X]/(X^2)$, $Z_2[X]/(X^3)$, $Z_3[X]/(X^2)$ or $Z_4[X]/(2X,X^2-2)$.

<u>Proof :</u> Since $\Gamma(R)$ is complete bipartite, then $\dim(\Gamma(R))=2$, and hence $R\cong F_1xF_2$ or R is a local ring Theorem 2.3. If $R\cong F_1xF_2$ we are done. If R local , then by Lemma 2.4 there exists $b\in Z(R)^*$ such that b is adjacent with every other vertices. So that $\Gamma(R)$ is star with center b. Therefore by Lemma 3.1 $R\cong A$. ■

<u>Theorem 3.3 :</u> Let R be a finite ring with $\Gamma(R)$ is complete bipartite, then

1- If R reduced , then $R\cong F_1xF_2$,and hence R is regular.
2- R satisfying condition (*). In particular, the only idempotent elements of R are 0,1,e,1-e .

<u>Proof :</u> (1) Since R reduced ring and $\Gamma(R)$ complete bipartite graph, then applying Proposition 3.2, $R\cong F_1xF_2$. Therefore R is regular ring,

(2) **<u>Claim 1:</u>** R satisfying (*) condition, where n=2.

Let $0,1,\neq e,f$ be any orthogonal idempotent elements in R and $\Gamma(R)$ complete bipartite, then we can write $\Gamma(R)=A\cup B$ with $A\cap B = \phi$ and for every two vertices from different partition class are adjacent and for every two vertices from same partition class are non-adjacent. Now if $e\in A$, then $f\in B$. Now since $e=e^2$, then $e(1-e)=0$ and hence $(1-e)\in B$. Similarly $1-f\in A$. Therefore $(1-e)(1-f)=0$ which implies that $e+f=1$ so that R satisfying condition (*) , where n=2 and f=1-e.

<u>Claim 2:</u> Let there exists idempotent elements $g^2=g\in R$ with $g\neq 0,1,e$ or (1-e) , since $\Gamma(R)$ complete bipartite graph , and $e(1-e)=0$, then either ge=0 or g(1-e) , without loss generality, let(1-g)e=0, then ge≠0 and so that g,e∈A or B. Let g,e∈A, then (1-g), (1-e)∈B and hence g(1-e)=e(1-g)=0, so that g=e which is a contradiction. Therefore the only idempotent elements in R is 0,1,e,1-e.

From Claims 1 and 2 , we conclude that R satisfying (*) condition ■

The condition " R is reduced "in Theorem 3.3(1) is not superfluous

Example 2: $R=Z_2[X]/(X^3)$ is not a reduced ring with Γ(R) complete bipartite graph and R not regular ring. ■

Proposition 3.4: Let R be a finite ring, then R is regular ring satisfying condition (*) if and only if R is reduced with Γ(R) is complete bipartite

Proof : Suppose that R is regular ring satisfying (*)condition , then obviously R is reduced and by Theorem 2.3, $\dim(\Gamma(R))\leq 2$ and by Theorem 2.1 we have $R\cong F_1xF_2$ which implies that Γ(R) is complete bipartite.

Conversely, assume R be a reduced ring with Γ(R) complete bipartite, then by Theorem 3.3, R is regular satisfying condition (*).■

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