An Approximate Solution of Non-Linear System of Volterra Integral Equation

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ABSTRACT

A Taylor series expansion is developed and applied to evaluate an approximate solution of the non-linear system of Volterra integral equation of the second kind for both Urysohn and Hammerstein types. The solution is based on substituting for the unknown function after differentiating both sides of the integral equation. Program associated with above methods is written in Matlab, finally, by using various examples, the accuracy of this method will be shown.

Keywords: non-linear system**,** Volterra integral equation.

الحل التقريبي للنظام غير الخطي لمعادلة فولتيرة التكاملية

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الملخص

طبقت متسلسلة تيلر لحساب الحل التقريبي لمنظومة غير خطية من معادلات فولتيرة التكاملية من الدرجة الثانية من نوع (Ursohn) و(Hammerstein) .الحل يعتمد على تعويض الدالة غير المعرفة بعد اشتقاق طرفي المعادلة التكاملية.جميع البرامج الخاصة بهذه الطريقة كتبت بلغة (Matlab) وتم استعراض النتائج بحل ثلاثة أمثلة اختباريه.

الكلمات المفتاحية: نظام غير خطي، معادلات فولتيرة.

1. Introduction

Integral equations appear in many engineering and physics, Numerical methods of solution for integral equations have been largely developed in the last 20 years [3,7]. Al-Faour used Taylor series expansion to evaluate the approximate solution of linear system of integral equations for Volterra type [2].

The main purpose of this paper is to consider Taylor series expansion of non-linear system of Volterra integral equation for Urysohn (SNLUVIEs) and Hammerstein (SNLHVIEs) types of the form

$$u_i(s) = f_i(s) + \sum_{j=1}^{n} \int_0^s K_{ij}(s,t,u_j(t))dt \qquad i = 1,2,...,n \qquad (1)$$

$$u_i(s) = f_i(s) + \sum_{j=1}^{n} \int_0^s Kij(s,t)W(t,u_j(t))dt \qquad i = 1,2,...,n \qquad (2)$$

where $k_{ij}(s,t)$ and $f_i(s)$ are known functions.

This system appear in many applications for instance: the Dirichelt-Neumann mixed boundary value problems (MBVPs) on closed surfaces in R^3 based on an equivalent formulation of the MBVP as a system of two integral equations [7].

2. Basic Theorem

Taylor theorem :

Assume that $y(t) \in C^{N+1}[t_0,b]$ and that y(t) has a Taylor series expansion of order N about the fixed value $t=t_k \in [t_0,b]$:

$$y(t_k + h) = y(t_k) + hT_N(t_k, y(t_k)) + O(h^{N+1}) \qquad (3)$$

where

$$T_N(t_k, y(t_k)) = \sum_{j=1}^{N} \frac{y^{(j)}(t_k)}{j!} h^{j-1} \qquad (4)$$

and $y^{(j)}(t) = f^{(j-1)}(t,y(t))$ denotes the (j-1)st total derivative of the function f with respect to t .

Fundamental Theorem of Integral Calculus (Leibnitz Generalized Formula):

$$\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} F(x,y)dy = \int_{\alpha(x)}^{\beta(x)} \frac{\partial F(x,y)}{\partial x} dy + F(x,\beta(x))\frac{d\beta(x)}{dx} - F(x,\alpha(x))\frac{d\alpha(x)}{dx}$$

Theorem (Precision of Taylor's Method of Order N):

Assume that y (t) is the solution to the initial value problem if $y(t) \in C^{N+1}[t_0,b]$ and $\{(t_k,y_k)\}_{k=0}^{M}$ is the sequence of approximations generated by Taylor's method of order N ,then

$$|e_k| = |y(t_k) - y_k| = O(h^{N+1})$$

$$|\in_{k+1}| = |y(t_{k+1}) - y_k - hT_N(t_k, y_k)| = O(h^N)$$

In particular, the final global error (F.G.E.) at the end of the interval will satisfy:

$$E(y(b),h)=\left|y(b)-Y_M\right|=O(h^N)$$

3. The Existence and Uniqueness Theorem

In this section we discuss the existence and uniqueness of the solution of equation (1) and (2).

Definition:

Let H be a Hilbert space and T a bounded on H, T is not necessary a linear operation, T is said to be a contraction operator if there exists a positive constant L<1 such that

$$\left\| Tf_1 - Tf_2 \right\| \le L \left\| f_1 - f_2 \right\| \qquad for all\ f_1, f_2\ in H$$

Theorem 1: (Fixed Point Theorem)

Let T be a contraction operator on H, and if

Tf = f

has a unique solution f in H, then such a solution is said to be a fixed point of T.(see [7] for proof)

Now in operator form eq. (1) can be written as:

$$U_m - K_m(U) = F_m, \qquad m = 1,2,...,n \tag{5}$$

where F_m , m=1,2,…,n are in H ,

$$K_m(U) = \sum_{j=1}^{n} \int_0^s K_{ij}\left(s,t,u_j(t)\right)dt$$

and K_m , m=1,2,…,n are bounded operators such that $\forall$ $M_m \in R^+$:

$$\left\| K_m(U_m) - K_m(V_m) \right\| \le M_m \left\| U_m - V_m \right\| \qquad , m = 1,2,...,n$$

Let M = max (M_1, M_2,…, M_m) , then we have

$$\left\| K_m(U) - K_m(V) \right\| \le M \left\| U - V \right\| \tag{6}$$

We can now write eq.(5) in the form :

$T(U_m) = U_m$, m=1,2,…,n (7)

where $T(U_m) = F_m + K_m(U)$, m=1,2,…,n

Theorem 2:

Equation (5) has a unique solution for all F_m , m=1,2,…,n , provided that K_m , m=1,2,…,n are bounded operators

4. Taylor series method for (SNLUVIEs)

Consider the following non-linear SUVIE :

$$u_i(s) = f_i(s) + \sum_{j=1}^{n} \int_0^s K_{ij}(s,t,u_j(t))dt \qquad i = 1,2,...,n \tag{8}$$

Differentiating both sides of eq. (7) 3-times with respect to s, to get

$$u_i'(s) = f_i'(s) + \sum_{j=1}^{n} \left\{ \int_a^s \frac{\partial K_{ij}\left(s,t,u_j(t)\right)}{\partial s} dt + K_{ij}\left(s,s,u_j(s)\right) \right\} \quad , \quad i = 1,2,...,n \tag{9}$$

$$u_i''(s) = f_i''(s) + \sum_{j=1}^{n} \left\{ \int_a^s \frac{\partial^2 K_{ij}\left(s,t,u_j(t)\right)}{\partial s^2} dt + \frac{\partial K_{ij}\left(s,t,u_j(s)\right)}{\partial s} + \frac{\partial K_{ij}\left(s,s,u_j(s)\right)}{\partial s} u_j'(s) \right\} \tag{10}$$

$$u_i'''(s) = f_i'''(s) + \sum_{j=1}^{n} \left\{ \begin{array}{l} \int_a^s \frac{\partial^3 K_{ij}\left(s,t,u_j(t)\right)}{\partial s^3} dt + \frac{\partial^2 K_{ij}\left(s,s,u_j(s)\right)}{\partial s^2} + \frac{\partial}{\partial s}\left\{ \frac{\partial K_{ij}\left(s,t,u_j(s)\right)}{\partial s} \right\} u_j'(s) + \\ \frac{\partial^2 K_{ij}\left(s,s,u_j(s)\right)}{\partial s^2} (u_j'(s))^2 + \frac{\partial K_{ij}\left(s,t,u_j(s)\right)}{\partial s} u_j''(s) \end{array} \right\} \tag{11}$$

Now, put s = a into eq. (8-11) to obtain:

$$u_i(a) = f_i(a) \qquad i = 1,2,...,n$$

$$u_i'(a) = f_i'(a) + \sum_{j=1}^{n} K_{ij}(a,a,u_j(a)) \quad , \quad i = 1,2,...,n$$

$$u_i''(a) = f_i''(a) + \sum_{j=1}^{n} \left\{ \frac{\partial K_{ij}(s,t,u_j(s)}{\partial s}\Big|_{t=s=a} + \frac{\partial K_{ij}\left(s,s,u_j(s)\right)}{\partial s}\Big|_{s=a} u_j'(a) \right\} \quad , i = 1,2,...,n$$

$$u_i'''(a) = f_i'''(a) + \sum_{j=1}^{n} \left\{ \begin{array}{l} \frac{\partial^2 K_{ij}(s,s,u_j(s))}{\partial s}\Big|_{s=a} + \frac{\partial}{\partial s}\left\{ \frac{\partial K_{ij}(s,t,u_j(s))}{\partial s} \right\}\Big|_{t=s=a} u_j'(a) \\ \frac{\partial^2 K_{ij}\left(s,s,u_j(s)\right)}{\partial s^2}\Big|_{s=a} (u_j'(a))^2 + \frac{\partial K_{ij}\left(s,t,u_j(s)\right)}{\partial s}\Big|_{t=s=a} u_j''(a) \ , \ i = 1,2,...,n \end{array} \right\}$$

5. Taylor series method for (NLSHVIE)

Consider the following non-linear SHVIE:

$$u_i(s) = f_i(s) + \sum_{j=1}^{n} \int_0^s Kij(s,t)W(t,u_j(t))dt \qquad i = 1,2,...,n \tag{12}$$

Differentiating both sides of eq. (12) 3 – times with respect to s, to get

$$u_i'(s)=f_i'(s)+\sum_{j=1}^{n}\left\{\int_a^s \frac{\partial K_{ij}(s,t)}{\partial s}W(t,u_j(t))\,dt+K_{ij}(s,s)W(s,u_j(s))\right\}\quad,\quad i=1,2,\ldots,n \tag{13}$$

$$u_i''(s)=f_i''(s)+\sum_{j=1}^{n}\left\{\begin{array}{l}\int_a^s \frac{\partial^2 K_{ij}(s,t)}{\partial s^2}W(t,u_j(t))\,dt+\frac{\partial K_{ij}(s,t)}{\partial s}W(s,u_j(s))+\frac{\partial K_{ij}(s,s)}{\partial s}W(s,u_j(s))\\ \qquad +K_{ij}(s,s)W'(s,u_j(s))\,u_j'(s)\end{array}\right\} \tag{14}$$

$$u_i'''(s)=f_i'''(s)+\sum_{j=1}^{n}\left\{\begin{array}{l}\int_a^s \frac{\partial^3 K_{ij}(s,t)}{\partial s^3}W(t,u_j(t))\,dt+\frac{\partial^2 K_{ij}(s,t)}{\partial s^2}\Big|_{t=s}W(s,u_j(s))+\frac{\partial}{\partial s}\left\{\frac{\partial K_{ij}(s,t)}{\partial s}\right\}\\ W(s,u_j(s))+\frac{\partial K_{ij}(s,t)}{\partial s}W'(s,u_j(s))u'(s)+\frac{\partial^2 K_{ij}(s,s)}{\partial s^2}W(s,u_j(s))\\ +\left(\frac{\partial K_{ij}(s,s)}{\partial s}W'(s,u_j(s))u_j'(s)\right)^2+K_{ij}(s,s)W''(s,u_j(s))\left(u_j'(s)\right)^2\\ K_{ij}(s,s)W'(s,u_j(s))u_j''(s)\qquad\qquad i=1,2,\ldots,n\end{array}\right\} \tag{15}$$

Now , put s = a into eqs.(12-15) to get :

$$u_i(a)=f_i(a)\quad,\quad i=1,2,\ldots,n$$

$$u_i'(a)=f_i'(a)+\sum_{j=1}^{n}K_{ij}(a,a)W(a,u_j(a))\quad,\quad i=1,2,\ldots,n$$

$$u_i''(a)=f_i''(a)+\sum_{j=1}^{n}\left\{\begin{array}{l}\frac{\partial K_{ij}(s,t)}{\partial s}\Big|_{t=s=a}W(a,u_j(a))+\frac{\partial K_{ij}(s,s)}{\partial s}\Big|_{s=a}W(a,u_j(a))\\ \qquad +K_{ij}(a,a)W'(a,u_j(a))\,u_j'(a)\end{array}\right\}$$

$$u_i'''(a)=f_i'''(a)+\sum_{j=1}^{n}\left\{\begin{array}{l}\frac{\partial^2 K_{ij}(s,t)}{\partial s^2}\Big|_{t=s=a}W(a,u_j(a))+\frac{\partial}{\partial s}\left\{\frac{\partial K_{ij}(s,s)}{\partial s}\Big|_{s=a}\right\}W\left(a,u_j(a)\right)\\ +\frac{\partial K_{ij}(s,t)}{\partial s}\Big|_{t=s=a}W'(a,u_j(a))u'(a)+\frac{\partial^2 K_{ij}(s,s)}{\partial s^2}\Big|_{s=a}W(a,u_j(a))\\ +2*\left(\frac{\partial K_{ij}(s,s)}{\partial s}\Big|_{s=a}W'(a,u_j(a))u_j'(a)\right)+K_{ij}(a,a)W''(a,u_j(a))\left(u_j'(a)\right)^2\\ K_{ij}(a,a)W'(a,u_j(a))u_j''(a)\qquad\qquad i=1,2,\ldots,n\end{array}\right\}$$

Solve the above system for the quantities $u_j'(a)$, $u_j''(a)$, and $u_j'''(a)$, j=1,2,…,n using forward substitution . Then these values are substituted in

$$u_j(s)=u_j(a)+u_j'(a)(s-a)+\frac{1}{2!}u_j''(a)(s-a)^2+\frac{1}{3!}u_j'''(a)(s-a)^3\quad,\quad j=1,2,\ldots,n$$

Taylor series eq. (3), (4) at $t = t_0$, N = 3, and $h = t-t_0$ to obtain the solution to order O (s^3)

6. Numerical Results

Here we present the results of applying the Taylor series expansion discussed to three different problems.

Example 1:

Consider the following non-linear system

$$u_1(s) = f_1 + \int_0^s (s-t)(u_2(t))^2\, dt$$

$$u_2(s) = f_2 + \int_0^s t\, e^{-2u_1(t)}\, dt$$

With the equation

$$f_1 = \frac{1}{4}(1-e^{2s}) \qquad \textit{and} \qquad f_2 = -se^s + 2e^s - 1$$

The exact solution of this problem is

$$u_1(s) = -\frac{1}{2}s \qquad \textit{and} \qquad u_2(s) = e^s$$

h = 0.1using a Taylor series expansion method with grid intervals 0< s <1 The approximate solution of the unknown function u(s) is given as:

$$u_1(s) = -\frac{1}{2}s \qquad \textit{and} \qquad u_2(s) = 1 + s + \frac{1}{2!}s^2 + \frac{1}{3!}s^3$$

A comparison of our results and exact solution gives on table (1).

s	$U_1(s)$		$U_2(s)$	
	Taylor	Exact	Taylor	Exact
0.0	-0.0000000	-0.0000000	1.0000000	1.0000000
0.1	-0.0500000	-0.0500000	1.1051666	1.1051709
0.2	-0.1000000	-0.1000000	1.2213333	1.2214028
0.3	-0.1500000	-0.1500000	1.3495000	1.3498588
0.4	-0.2000000	-0.2000000	1.4906666	1.4918247
0.5	-0.2500000	-0.2500000	1.6458333	1.6487213
0.6	-0.3000000	-0.3000000	1.8160000	1.8221188
0.7	-0.3500000	-0.3500000	2.0021666	2.0137527
0.8	-0.4000000	-0.4000000	2.2053333	2.2255409
0.9	-0.4500000	-0.4500000	2.4265000	2.4596031
1.0	-0.5000000	-0.5000000	2.6666666	2.7182818
L.S.E.	0.00000000		$4.3491*10^{-3}$	

Table (1)

Example 2:

Given $f_1(s)$, $f_2(s)$, We wish to find $\Phi_1(s)$, $\Phi_2(s)$ so that

$$\Phi_1(s) = f_1(s) + \int_0^s \ln \left| \Phi_2(t) \right| dt$$

$$\Phi_2(s) = f_2(s) + \int_0^s t\, e^{\Phi_1(t)}\, dt$$

With the functions

$$f_1(s) = s(1 - \frac{1}{2} s) \qquad \textit{and} \qquad f_2(s) = -(s\, e^s + 1)$$

The exact solution is:

$$\Phi_1(s) = s \qquad \textit{and} \qquad \Phi_2(s) = -e^s$$

When we apply Taylor's method, we have the following approximate solution

$$\Phi_1(s) = s \qquad \textit{and} \qquad \Phi_2(s) = -1 - s - \frac{1}{2!} s^2 - \frac{1}{3!} s^3$$

Table (2) gives a comparison of our results and exact solution

S	$\Phi_1(s)$		$\Phi_2(s)$	
	Taylor	Exact	Taylor	Exact
0.0	0.0000000	0.0000000	-1.0000000	-1.0000000
0.1	0.1000000	0.1000000	-1.1051666	-1.1051709
0.2	0.2000000	0.2000000	-1.2213333	-1.2214028
0.3	0.3000000	0.3000000	-1.3495000	-1.3498588
0.4	0.4000000	0.4000000	-1.4906666	-1.4918247
0.5	0.5000000	0.5000000	-1.6458333	-1.6487213
0.6	0.6000000	0.6000000	-1.8160000	-1.8221188
0.7	0.7000000	0.7000000	-2.0021666	-2.0137527
0.8	0.8000000	0.8000000	-2.2053333	-2.2255409
0.9	0.9000000	0.9000000	-2.4265000	-2.4596031
1.0	1.0000000	1.0000000	-2.6666666	-2.7182818
L.S.E.	0.00000000		$4.3491*10^{-3}$	

Table (2)

Example 3:

As a third example, we study the following non-linear differential equation of the second order

$$y'' + \sin(x)\ (\frac{\cos(x)}{y'} - \frac{1}{y}) = -1, \quad x \in (0,1), \quad y(0) = 0, \quad y'(0) = 1$$

Which can be written as a system of first order non-linear differential equations

$$y_1' = y_2 \qquad y_1(0) = 0$$

$$y_2' = \sin(x)\left(\frac{1}{y_1} - \frac{\cos(x)}{y_2}\right) - 1 \qquad y_2(0) = 1 \qquad (16)$$

where the exact solution of this problem is:
$y_1 = \sin(x)$ and $y_2 = \cos(x)$
by integrating both sides of eq.(16) over [0,x], we obtained the following NLSVIEs

$$y_1(x) = \int_0^x y_2(t)\,dt$$

$$y_2(x) = 1 - x + \int_0^x \frac{\sin(t)}{y_1(t)}\,dt - \int_0^x \frac{\sin(t)\cos(t)}{y_2(t)}\,dt$$

Table (3) gives a comparison of our results and exact solution.

x	y_1(s)		y_2(s)	
	Taylor	Exact	Taylor	Exact
0.0	0.000000	0.000000	1.000000	1.000000
0.1	0.099833	0.099833	0.995004	0.995004
0.2	0.198667	0.198666	0.980066	0.980066
0.3	0.295500	0.295520	0.955336	0.955337
0.4	0.389333	0.389418	0.921061	0.921066
0.5	0.479167	0.479425	0.877583	0.877604
0.6	0.564000	0.564425	0.825336	0.825400
0.7	0.642833	0.644218	0.764842	0.765004
0.8	0.714667	0.717356	0.696707	0.697066
0.9	0.778500	0.783327	0.621609	0.622337
1.0	0.833333	0.841471	0.540302	0.541666
L.S.E.	$90958 * 10^{-3}$		$2.545*10^{-6}$	

Table (3)

7-Conclusion

Taylor series expansion method was used to evaluate an approximate solution of non-linear system of Volterra Integral Equation for both Urysohn and Hammerstein types. Three examples were considered in this context.
In practice, we conclude that:

The solution obtained by Taylor series expansion is given by a function not only at some points.

Numerical computations of Taylor's method are simple and the convergence is satisfactory.

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