Stability Conditions of Zero Solution for Third Order Differential Equation in Critical Case

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ABSTRACT

In this paper, we study the conditions under which the zero solution is stable in the semi- liner case for certain third order differential equation of the form :

$y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = h(t, y, y', y'')$

Where

$p_s = \pi^s[q_s + w_s(t)]$, s = 1,2,3 $w_s(t): \Delta \to C$, $q_s \in C$,
$a \in N$, $t \in \Delta = [a, \infty)$

The characteristic equation of the above differential equation has complex roots of the form :
and the other root has the following $\lambda_0 > 0$, $\lambda_1 = -\lambda_2 = i\lambda_0$
. $\text{Re}\,\lambda_3 < -M, M > 0$ property

Keywords: Stability, Critical Case.

شروط استقرارية الحل الصفري لمعادلة تفاضلية من الرتبة الثالثة في احدى الحالات الحرجة

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الملخص

هذا البحث سيدرس شروط استقرارية الحل الصفري في الحالة شبه الخطية لمعادلة تفاضلية من الرتبة الثالثة بالشكل : $y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = h(t, y, y', y'')$

حيث ان

$p_s = \pi^s[q_s + w_s(t)]$, s = 1,2,3 $w_s(t): \Delta \to C$, $q_s \in C$,
$a \in N$, $t \in \Delta = [a, \infty)$

ان المعادلة المميزة للمعادلة التفاضلية اعلاه لها زوج من الجذور المعقدة بالشكل :
$\lambda_0 > 0$, $\lambda_1 = -\lambda_2 = i\lambda_0$

والجذر الاخر يحقق الخاصية $\text{Re}\,\lambda_3 < -M, M > 0$.

الكلمات المفتاحية: الاستقرارية, الحالة الحرجة

1- INTRODUCTION

Critical cases in the theory of stability for differential equation means , that cases when the real part of all roots of the characteristic equation are nonpositive with the real part of at least one root being zero , other express which is neither stable nor unstable [3] .

In the critical case the non-liner terms begin to influence the stability of a stationary point and the investigation of the first approximation for stability is in general impossible .

In [4,5] studied the conditions of stability zero solution for certain differential equation in the semi-linear case when the characteristic equation has roots of the form : $\lambda_1 = i\lambda_0 , \lambda_0 > 0$

and the others satisfying the property $\operatorname{Re}\lambda_k < -M, M > O, k = 2,....., n$.

[4,6] studied the same conditions to find the center of gravity for nonoutonomous quasi-linear differential equation of n-th order .

In this paper, we study the conditions under which the zero solution is stable in the semi–linear case of differential equation which has the form :

$$y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = h(t, y, y', y'') \quad \ldots(1)$$

Where

s = 1,2,3, $w_s(t) : \Delta \to C$, $q_s \in C$, $p_s = \pi^s[q_s + w_s(t)]$

$t \in \Delta = [a, \infty)$, $a \in N$

are continuos functions for all s and $\pi^s(t)$ times differentiable and α satisfies the following conditions :

, $\pi^{-2}\pi' = 0(1)$, $\pi : \Delta \to (0, \infty)$ $t \longrightarrow \infty$

$$|h(t, y, y', y'')| \le L^*\left[|y| + |y'| + |y''|\right]^{1+\beta} , h : \Delta \times C^n \to C$$

$L^* : \Delta \to [0, \infty)$, $\beta \ge 0$ and the characteristic equation of (1) has roots, $\lambda_1 = -\lambda_2 = i\lambda_0 , \lambda_0 > 0$ and the other root has the following property $\operatorname{Re}\lambda_3 < -M, M > 0$.

2- Definitions :

Definition 1 [3] : The zero solution of the differential equation (1) is said to be stable as such that $\delta > 0$ there exist $\forall \varepsilon > 0$, if $t \longrightarrow \infty$ the solution y = y(t) of the differential equation (1) with the initial condition $|y(T)| < \delta$ satisfies the inequality $|y(t)| < \varepsilon$, $\forall t \ge T$

Definition 2 [3]: If the conditions of definition(1) are satisfied and then , zero solution of (1) is said to be asymptotically $\lim_{t\to\infty} x(t) = 0(1)$ stable .

3 – Helping Transformations

In order to find the conditions under which the zero solution of differential equation (1) is stable , we use the following lemmas :

Lemma 1 [2] :

The transformation ;

$$\left.\begin{aligned} y &= \pi \, . \, Z_1 \\ y' &= \pi^2 . \, Z_2 \\ y'' &= \pi^3 . \, Z_3 \end{aligned}\right\} \quad \ldots(2)$$

transform the differential equation (1) to the differential system of the form :

$$\left.\begin{aligned} Z_1' &= -\pi^{-1}\pi' Z_1 + \pi Z_2 \\ Z_2' &= -2\pi^{-1}\pi' Z_2 + \pi Z_3 \\ Z_3' &= -p_3(t)\pi^{-2} Z_1 - p_2(t)\pi^{-1} Z_2 - (3\pi^{-1}\pi' + p_1(t))\, Z_3 + F_1(t,z) \\ &\text{where} \\ |F_1(t\,,\,z)| &\le L\,[Z_1 + \pi Z_2 + \pi^2 Z_3]^{1+\beta},\ L = L^* \pi^{\beta-2} \end{aligned}\right\} \quad \ldots(3)$$

Lemma 2 [2] :

The transformation ;

X = BZ …(4)

where

$$B=\begin{bmatrix}(i\lambda_0)^2+q_1(i\lambda_0)+q_2 & i\lambda_0+q_1 & 1\\ (-i\lambda_0)^2+q_1(i\lambda_0)+q_2 & -i\lambda_0+q_1 & 1\\ 1 & 0 & 0\end{bmatrix}$$

$q_1,q_2 \in C$, det B = $2i\lambda_0$

transform the differential system (3) to the system of the form :

$$\left.\begin{aligned} X_1' &= \pi[i\lambda_0 X_1+(-(i\lambda_0)^3-q_1(i\lambda_0)^2-q_2 i\lambda_0-q_3)X_3]+F_2\\ X_2' &= \pi[q_1X_1+(-i\lambda_0-q_1)X_2+((i\lambda_0)^3+(i\lambda_0)^2q_1+i\lambda_0q_2-q_3)X_3]+F_2\\ X_3' &= \pi[\frac{1}{2i\lambda_0}X_1-\frac{1}{2i\lambda_0}X_2]\end{aligned}\right\} \quad \ldots(5)$$

where , $\lim_{t\to\infty} W_s(t)=o(1)$, $\lim_{t\to\infty}\pi^{-2}\pi'=o(1)$

$$|F2|\le ML\left[\frac{[-\pi+\pi^2(-i\lambda_o+q_1)]X_1+[\pi-\pi^2(i\lambda_0+q_1)]X_2+[-2i\lambda_0+\pi^2 2(i\lambda_0)^3+2q_1(i\lambda_0)^2+2q_2 i\lambda_0)]X_3}{-2i\lambda_0}\right]^{1+\beta}$$

$M>0$

Lemma 3 [2] :

By using the following transformation

$$\left.\begin{aligned} X_1 &= y_1\\ X_2 &= y_2\\ X_3 &= ky_1+\bar{k}y_2+y_3\end{aligned}\right\} \quad \ldots(6)$$

where k , $\bar{k} \in C$

we transform the differential system (5) into the following differential system :

$$\left.\begin{aligned} y_1' &= \pi[(i\lambda_0+c_1k)y_1+(c_1\bar{k})y_2+c_1y_3]+F_3\\ y_2' &= \pi[(q_1+c_2k)y_1+(-i\lambda_0-q_1+c_2\bar{k})y_2+c_2y_3]+F_3\\ y_3' &= \pi\left[[\frac{1}{2i\lambda_0}-k(i\lambda_0+c_1k)-\bar{k}(q_1+c_2k)]y_1+[-\frac{1}{2i\lambda_0}-kc_1\bar{k}-\bar{k}(-i\lambda_0-q_1+c_2\bar{k})]y_2+\right.\\ &\qquad [-kc_1-\bar{k}c_2]y_3\Big]-F_3[k+\bar{k}]\end{aligned}\right\} \quad \ldots(7)$$

$$|F3| \le ML\left[\frac{(-\pi+\pi^2(-i\lambda_0+q_1)+k[-2i\lambda_0+\pi^2(2(i\lambda_0)^3+2q_1(i\lambda_0)^2+2q_2 i\lambda_0)}{-2i\lambda_0}y_1+\right.$$

$$\frac{(\pi-\pi^2(-i\lambda_0+q_1)+\bar{k}[-2i\lambda_0+\pi^2(2(i\lambda_0)^3+2q_1(i\lambda_0)^2+2q_2 i\lambda_0)]}{-2i\lambda_0}y_2$$

$$\left.+[\frac{-2i\lambda_0+\pi^2(2(i\lambda_0)^3+2q_1(i\lambda_0)^2+2q_2 i\lambda_0)}{-2i\lambda_0}]y_3\right]^{1+\beta}$$

Now , we use the following lemma which leads to the auxiliary system :

Lemma 4 [5] :

the transform

$$\left.\begin{aligned} y_1 &= w_1+bw_2+b_3w_3 \\ y_2 &= bw_1+w_2+b_3w_3 \\ y_3 &= w_3 \end{aligned}\right\} \qquad \ldots(8)$$

where $b, b_3 \in C$, $b \neq \pm 1$

$$w_1' = \frac{\pi}{1-b^2}\Big[i\lambda_0+c_1k-b(q_1+c_2k)+b_3(b-1)[\frac{1}{2i\lambda_0}-k(i\lambda_0+c_1k)$$

$$-\bar{k}(q_1+c_2k)]+b[c_1\bar{k}-b(-i\lambda_0-q_1+c_2\bar{k})+b_3(b-1)$$

$$[-\frac{1}{2i\lambda_0}-kc_1\bar{k}-\bar{k}(-i\lambda_0-q_1+c_2\bar{k})]]\Big]W_1+$$

$$\frac{\pi}{1-b^2}\Big[b[i\lambda_0+c_1k-b(q_1+c_2k)+b_3(b-1)[\frac{1}{2i\lambda_0}-k(i\lambda_0+c_1k)$$

$$-\bar{k}(q_1+c_2k)]]+c_1\bar{k}-b(-i\lambda_0-q_1+c_2\bar{k})+b_3(b-1)[-\frac{1}{2i\lambda_0}-kc_1\bar{k}-$$

$$\bar{k}(-i\lambda_0-q_1+c_2\bar{k})]\Big]W_2+$$

$$+\frac{\pi}{1-b^2}\Big[b_3[i\lambda_0+c_1k-b(q_1+c_2k)+b_3(b-1)[-k(i\lambda_0+c_1k)-\bar{k}(q_1+c_2k)]+c_1\bar{k}-$$

$$b(-i\lambda_0-q_1+c_2\bar{k})+b_3(b-1)[-kc_1\bar{k}-\bar{k}(-i\lambda_0-q_1+c_2k)]]+c_1-bc_2+b_3(b-1)$$

$$[-kc_1-\bar{k}c_2]\Big]W_3+\frac{1}{1-b^2}F_4[1-b-b_3(b-1)(k+\bar{k})]$$

□

$$
\left.\begin{aligned}
w_2' &= \frac{\pi}{1-b^2}\Big[-b(i\lambda_0+c_1k)+q_1+c_2k+b_3(b-1)[\frac{1}{2i\lambda_0}-k(i\lambda_0+c_1k) \\
&-\bar{k}(q_1+c_2k)]+b[-bc_1\bar{k}-i\lambda_0-q_1+c_2\bar{k}+b_3(b-1) \\
&[-\frac{1}{2i\lambda_0}-kc_1\bar{k}-\bar{k}(-i\lambda_0-q_1+c_2\bar{k})]]\Big]W_1+ \\
&\frac{\pi}{1-b^2}\Big[b[-b(i\lambda_0+c_1k)+q_1+c_2k+b_3(b-1)[\frac{1}{2i\lambda_0}-k(i\lambda_0+c_1k) \\
&-\bar{k}(q_1+c_2k)]]-bc_1\bar{k}-i\lambda_0-q_1+c_2\bar{k}+b_3(b-1)[-\frac{1}{2i\lambda_0}-kc_1\bar{k}- \\
&\bar{k}(-i\lambda_0-q_1+c_2\bar{k})]\Big]W_2 \\
&+\frac{\pi}{1-b^2}\Big[b_3[-b(i\lambda_0+c_1k)+q_1+c_2k+b_3(b-1)[-k(i\lambda_0+c_1k) \\
&-\bar{k}(q_1+c_2k)]-bc_1\bar{k}-i\lambda_0-q_1+c_2\bar{k}+b_3(b-1)[-kc_1\bar{k}- \\
&\bar{k}(-i\lambda_0-q_1+c_2\bar{k})]-bc_1+c_2+b_3(b-1)[-kc_1-\bar{k}c_2]\Big]W_3+ \\
&+\frac{1}{1-b^2}F_4[1-b-b_3(b-1)(k+\bar{k})] \\
w_3' &= \pi\Big[\frac{1}{2i\lambda_0}-k(i\lambda_0+c_1k)-\bar{k}(q_1+c_2k)+b(-\frac{1}{2i\lambda_0}-kc_1\bar{k}-\bar{k}(-i\lambda_0-q_1+c_2\bar{k}))\Big]W_1+ \\
&+\pi\Big[b(\frac{1}{2i\lambda_0}-k(i\lambda_0+c_1k)-\bar{k}(q_1+c_2k))-\frac{1}{2i\lambda_0}-kc_1\bar{k}-\bar{k}(-i\lambda_0-q_1+c_2\bar{k})\Big]W_2 \\
&+\pi\Big[b_3[-k(i\lambda_0+c_1k)-\bar{k}(q_1+c_2k)-kc_1\bar{k}-\bar{k}(-i\lambda_0-q_1+c_2\bar{k})]-kc_1-\bar{k}c_2\Big]w_3-F_4(k+\bar{k})
\end{aligned}\right\}\quad\ldots(9)
$$

$$|F4| \le ML \left[\frac{1}{-2i\lambda_o}[-\pi+\pi^2(-i\lambda_o+q_1)+k[-2i\lambda_o+\pi^2(2(i\lambda_0)^3+2q_1(i\lambda_0)^2+2q_2 i\lambda_0)] \right.$$

$$+b[\pi-\pi^2(i\lambda_o+q_1)+\bar{k}[-2i\lambda_0+\pi^2(2(i\lambda_0)^3+2q_1(i\lambda_0)^2+2q_2 i\lambda_0)]]]W_1$$

$$+\frac{1}{-2i\lambda_0}[b[-\pi+\pi^2(-i\lambda_0+q_1)+k[-2i\lambda_0+\pi^2(2(i\lambda_0)^3+2q_1(i\lambda_0)^2+2q_2 i\lambda_0)]]$$

$$+\pi-\pi^2(i\lambda_0+q_1)+\bar{k}[-2i\lambda_0+\pi^2(2(i\lambda_0)^3+2q_1(i\lambda_0)^2+2q_2 i\lambda_0)]]W_2$$

$$+\frac{1}{-2i\lambda_0}[b_3[-2i\lambda_0\pi^2+k[-2i\lambda_0+\pi^2(2(i\lambda_0)^3+2q_1(i\lambda_0)^2+2q_2 i\lambda_0)]$$

$$\left. +\bar{k}[-2i\lambda_0+\pi^2(2(i\lambda_0)^3+2q_1(i\lambda_0)^2+2q_2 i\lambda_0)]]-2i\lambda_0+\pi^2(2(i\lambda_0)^3+2q_1(i\lambda_0)^2+2q_2 i\lambda_0\ ]W_3 \right]^{1+\beta}$$

Lemma 5 [2] :

the transform

$$\left.\begin{aligned} w_1 &= re^{i\theta} \\ w_2 &= re^{-i\theta} \\ w_3 &= -r_3 \end{aligned}\right\} \qquad \ldots(10)$$

where $\theta \in [0,2\pi]$

transform (9) into the following differential system :

$$\left.\begin{aligned} r' &= \mu_1 r+\mu_2 r_3+\frac{e^{-i\theta}}{1-b^2}F_5[1-b-b_3(b-1)(k+\bar{k})] \\ r' &= \mu_1^* r+\mu_2^* r_3+\frac{e^{i\theta}}{1-b^2}F_5[1-b-b_3(b-1)(k+\bar{k})] \\ r_3' &= \mu_1^{**} r+\mu_2^{**} r_3-F_5[k+\bar{k}] \end{aligned}\right\} \qquad \ldots(11)$$

where

$$\mu_1 = \frac{\pi}{1-b^2}[i\lambda_0 + c_1 k - b(q_1 + c_2 k) + b_3(b-1)[\frac{1}{2i\lambda_0} - k(i\lambda_0 + c_1 k)$$
$$-\bar{k}(q_1 + c_2 k)] + b[c_1\bar{k} - b(-i\lambda_0 - q_1 + c_2\bar{k}) + b_3(b-1)[-\frac{1}{2i\lambda_0} - kc_1\bar{k}$$
$$-\bar{k}(-i\lambda_0 - q_1 + c_2\bar{k})]]$$
$$+\frac{\pi}{1-b^2}[b[i\lambda_0 + c_1 k - b(q_1 + c_2 k) + b_3(b-1)[\frac{1}{2i\lambda_0} - k(i\lambda_0 + c_1 k)$$
$$-\bar{k}(q_1 + c_2 k)]] + c_1\bar{k} - b(i\lambda_0 - q_1 + c_2\bar{k}) + b_3(b-1)[-\frac{1}{2i\lambda_0} - kc_1\bar{k} -$$
$$\bar{k}(-i\lambda_0 - q_1 + c_2\bar{k})]]e^{-2i\theta}$$
$$\mu_2 = -\frac{\pi e^{-i\theta}}{1-b^2}b_3[i\lambda_0 + c_1 k - b(q_1 + c_2 k) + b_3(b-1)[-k(i\lambda_0 + c_1 k) - \bar{k}(q_1 +$$
$$c_2 k)] + c_1\bar{k} - b(-i\lambda_0 - q_1 + c_2\bar{k}) + b_3(b-1)[-kc_1\bar{k} - \bar{k}(-i\lambda_0 - q_1 + c_2\bar{k})]]$$
$$-\frac{\pi e^{-i\theta}}{1-b^2}[c_1 - bc_2 + b_3(b-1)[-kc_1 - \bar{k}c_2]].$$
$$\mu_1^* = \frac{\pi}{1-b^2}[-b(i\lambda_0 + c_1 k) + q_1 + c_2 k + b_3(b-1)[\frac{1}{2i\lambda_0} - k(i\lambda_0 + c_1 k)$$
$$-\bar{k}(q_1 + c_2 k)] + b[-bc_1\bar{k} - i\lambda_0 - q_1 + c_2\bar{k} + b_3(b-1)[-\frac{1}{2i\lambda_0} - kc_1\bar{k}$$
$$-\bar{k}(-i\lambda_0 - q_1 + c_2\bar{k})]]e^{2i\theta}$$
$$+\frac{\pi}{1-b^2}[b[-b(i\lambda_0 + c_1 k) + q_1 + c_2 k + b_3(b-1)[\frac{1}{2i\lambda_0} - k(i\lambda_0 + c_1 k)$$
$$-\bar{k}(q_1 + c_2 k)]] - bc_1\bar{k} - i\lambda_0 - q_1 + c_2\bar{k} + b_3(b-1)[-\frac{1}{2i\lambda_0} - kc_1\bar{k} -$$
$$\bar{k}(-i\lambda_0 - q_1 + c_2\bar{k})]]$$

$$\mu_2^* = -\frac{\pi e^{i\theta}}{1-b^2} b_3[-b(i\lambda_0 + c_1 k) + q_1 + c_2 k + b_3(b-1)[-k(i\lambda_0 + c_1 k) - \bar{k}(q_1 +$$

$$c_2 k)] - bc_1 \bar{k} - i\lambda_0 - q_1 + c_2 \bar{k} + b_3(b-1)[-kc_1 \bar{k} - \bar{k}(-i\lambda_0 - q_1 + c_2 \bar{k})]]$$

$$-\frac{\pi e^{i\theta}}{1-b^2}[-bc_1 + c_2 + b_3(b-1)[-kc_1 - c_2 \bar{k}]]$$

$$\mu_1^{**} = -\pi[\frac{1}{2i\lambda_0} - k(i\lambda_0 + c_1 k) - \bar{k}(q_1 + c_2 k) + b[-\frac{1}{2i\lambda_0} - kc_1 \bar{k} -$$

$$-\bar{k}(-i\lambda_0 - q_1 + c_2 \bar{k})]]e^{i\theta}$$

$$-\pi[b[\frac{1}{2i\lambda_0} - k(i\lambda_0 + c_1 k) - \bar{k}(q_1 + c_2 k)] - \frac{1}{2i\lambda_0} - kc_1 \bar{k} - \bar{k}(-i\lambda_0 - q_1 + c_2 \bar{k})]]e^{-i\theta}$$

$$\mu_2^{**} = -\pi b_3[-k(i\lambda_0 + c_1 k) - \bar{k}(q_1 + c_2 k) - kc_1 \bar{k} - \bar{k}(-i\lambda_0 - q_1 + c_2 \bar{k})]$$

$$-\pi[-kc_1 - \bar{k} c_2]$$

$$|F5| \le ML\left[\frac{1}{-2i\lambda_0}[-\pi + \pi^2(-i\lambda_0 + q_1) + k[-2i\lambda_0 + \pi^2(2(i\lambda_0)^3 + 2q_1(i\lambda_0)^2 + 2q_2 i\lambda_0)]\right.$$

$$+b[\pi - \pi^2(i\lambda_0 + q_1) + \bar{k}[-2i\lambda_0 + \pi^2(2(i\lambda_0)^3 + 2q_1(i\lambda_0)^2 + 2q_2 i\lambda_0)]]]e^{i\theta}$$

$$+[b[-\pi + \pi^2(i\lambda_0 + q_1) + k[-2i\lambda_0 + \pi^2(2(i\lambda_0)^3 + 2q_1(i\lambda_0)^2 + 2q_2 i\lambda_0)]]]$$

$$+\pi - \pi^2(i\lambda_0 + q_1) + \bar{k}[-2i\lambda_0 + \pi^2(2(i\lambda_0)^3 + 2q_1(i\lambda_0)^2 + 2q_2 i\lambda_0)]]e^{i\theta})r$$

$$-\frac{1}{-2i\lambda_0}[b_3[-2i\lambda_0\pi^2 + [k + \bar{k}](-2i\lambda_0 + \pi^2(2(i\lambda_0)^3 + 2q_1(i\lambda_0)^2 + 2q_2 i\lambda_0))]$$

$$\left.-2i\lambda_0 + \pi^2(2(i\lambda_0)^3 + 2q_1(i\lambda_0)^2 + 2q_2 i\lambda_0]r_3\right]^{1+\beta}$$

4 – Fundamental results :

Theorem :

In the equation (1) if :

$, h : \Delta \times C^n \to C$, $p_s : \Delta = [a, \infty) \to C$ 1-

$|h(t, y, y', y'')| \le L^*[|y| + |y'| + |y''|]^{1+\beta}$

$L^* : \Delta \to [0, \infty)$, $\beta \ge 0$

2 - $\lim_{t\to\infty} \pi^{-2}\pi' = o(1)$, $\lim_{t\to\infty} W_s(t) = o(1)$

and

a – $t \to \infty$ as $\int_T^t \operatorname{Re} \mu dt \to -\infty$ if

b - $e^{\int_T^t \operatorname{Re} \mu dt} \int_T^t \mu_1 e^{-\int_T^t \operatorname{Re} \mu dt} dt = o(1), t \to \infty$

then the zero solution of (1) is stable

c – as $\int_T^t \operatorname{Re} \mu dt \to \infty$ if $t \to \infty$

then the zero solution of (1) is unstable

proof

on applying the transformation (2) ,(4) ,(8) and (10) into (1) we get the auxiliary system :

$$r' = \mu_1 r + \mu_2 \xi_3 + \frac{e^{-i\theta}}{1-b^2} F_6 \left[1 - b - b_3(b-1)(k+\bar{k})\right]$$

$$r' = \mu_1^* \xi_2 + \mu_2^* \xi_2 + \frac{e^{-i\theta}}{1-b^2} F_6 \left[1 - b - b_3(b-1)(k+\bar{k})\right] \quad \text{….(12)}$$

$$r_3' = \mu_1^{**} \xi_1 + \mu_2^{**} r_3 - F_6 \left[k + \bar{k}\right]$$

is an arbitrary variant function and it is continuous $\xi = \xi(t)$ where ,
for all " $t \geq T$ the auxiliary system (12) solved by the method Variation of parameters " [1]

$$|r| \leq e^{\int_T^t \operatorname{Re} \mu_1 dt} [r(T) + \int_T^t (\mu_2 \xi_3 + \frac{e^{-i\theta}}{1-b^2} F_6 [1 - b - b_3(b-1)(k+\bar{k})]) e^{-\int_T^t \operatorname{Re} \mu_1(t) dt} dt] \quad \text{….(13)}$$

$$|r| \leq e^{\int_T^t \operatorname{Re} \mu_1^* dt} [r(T) + \int_T^t (\mu_2^* \xi_2 + \frac{e^{i\theta}}{1-b^2} F_6 [1 - b - b_3(b-1)(k+\bar{k})]) e^{-\int_T^t \operatorname{Re} \mu_1^*(t) dt} dt] \quad \text{….(14)}$$

$$|r_3| \leq e^{\int_T^t \operatorname{Re} \mu_2^{**} dt} [r(T) + \int_T^t (\mu_1^{**} \xi_1 - F_6 [(k+\bar{k})] e^{-\int_T^t \operatorname{Re} \mu_2^{**}(t) dt} dt] \quad \text{…..(15)}$$

Now it is clear if

1- $e^{\int_T^t \operatorname{Re} \mu dt} = 0$, 2 - $e^{\int_T^t \operatorname{Re} \mu^* dt} = 0$, 3 - $e^{\int_T^t \operatorname{Re} \mu^{**} dt} = 0$

then the zero solution of equation (1) is stable

To explain our fundamental results the following example is given:

$$y''' + \pi[(1-i)+(\ln t)^{-1}(2+3i)]y'' + \pi^2[4+(\ln t)^{-2}(1-4i)]y'+\pi^3[(4-4i)+t^{-1}(3-6i)]y = L^*[|y|+|y'|+|y''|]^{1+\beta} \quad \text{.............(A)}, \qquad \beta\in[0,\infty)$$

The characteristic equation to homogeneous part of equation (A)

Contains roots form : $\lambda_1 = 2i$, $\lambda_2 = -2i$, $\lambda_3 = -1+ i$

when $\lim_{t\to\infty} \pi^{-2}\pi' = o(1)$, $\lim_{t\to\infty} W_s(t) = o(1)$

$b = 1+i$, $b_3 = 1$, $k = 2+i$, $\bar{k} = 2 - i$

then $\pi = -t^{-1/2}$ or $\pi = e^t$ for example if $\lim_{t\to\infty} \pi^{-2}\pi' = o(1)$

on applying the transformation (2) ,(4) ,(8) and (10) into (A) we get the following table :

	$\pi = -t^{-1/2}$				**$\pi = e^t$**			
	θ=0	**θ=30**	**θ=45**	**θ=60**	**θ=0**	**θ=30**	**θ=45**	**θ=60**
$\int_T^t \operatorname{Re}\mu_1\, dt$	∞	- ∞	- ∞	- ∞	- ∞	∞	∞	∞
$\int_T^t \operatorname{Re}\mu_1^*\, dt$	∞	∞	∞	∞	- ∞	- ∞	- ∞	- ∞
$\int_T^t \operatorname{Re}\mu_2^{**}\, dt$	∞				- ∞			
	unstable				stable	unstable	unstable	unstable

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