On Centrally Prime and Centrally Semiprime Rings

Adil. K. Jabbar
Mathematics Department
College of Science
University of Sulaimani

Abdularahman. H. Majeed
Mathematics Department
College of Science
University of Baghdad

Received on: 14/08/2006 **Accepted on: 24/12/2006**

ABSTRACT

In this paper, centrally prime and centrally semiprime rings are defined and the relations between these two rings and prime (resp. semiprime) rings are studied.Among the results of the paper some conditions are given under which prime (resp. semiprime) rings become centrally prime (resp.centrally semiprime) as in:1-A nonzero prime (resp. semiprime) ring which has no proper zero divisors is centrally prime (resp.centrally semiprime).Also we gave some other conditions which make prime (resp. semiprime) rings and centrally prime (resp.centrally semiprime) rings equivalent, as in :2-A ring which satisfies the-(BZP) for multiplicative systems is prime (resp. semiprime) if and only if it is centrally prime (resp.centrally semiprime).3-A ring with identity in which every nonzero element of its center is a unit is prime (resp. semiprime) if and only if it is centrally prime (resp.centrally semiprime).

Keywords: prime rings, semiprime rings, centrally prime rings, centrally semiprime rings, localization.

حول الحلقات الأولية مركزيا والحلقات شبه الأولية مركزيا

عادل جبار
قسم الرياضيات
كلية العلوم
جامعة السليمانية

عبد الرحمن حميد مجيد
قسم الرياضيات
كلية العلوم
جامعة بغداد

تاريخ الاستلام: 2006/8/14 **تاريخ القبول: 2006/12/24**

الملخص

فى هذا االبحث قدمت تعريفالحلقا ت الاولية مركزيا والحلقا ت شبه الاولية مركزيا وتمت دراسة العلاقات بين هاتين الحلقتين والحلقات الاولية وشبه الاولية. من بين النتائج التى توصلنا اليها هى تحديد بعض الشروط التى عند توافرها تصبح الحلقات الاولية (شبه الاولية) حلقات اولية (شبه اولية) مركزيا كما فى: 1 –الحلقات الاولية (شبه الاولية)غير الصفرية التى ليست لها قواسم صفرية فعلية تكون حلقات اولية(شبه اولية)مركزيا.

كذلك اعطينا شروطا اخرى تجعل من الحلقات الاولية(شبه الاولية) والحلقات الاولية(شبه الاولية)مركزيا حلقات متكافئة كما فى:2 –الحلقات التى تحقق(خاصية –ذات الصفرين–) للانظمة الضربية تكون حلقات اولية(شبه اولية) اذا وفقط اذا كانت حلقات اولية(شبه اولية) مركزيا. 3–الحلقة ذات العنصر المحايدالضربى التى تشكل كل عنصر غير صفرى من عناصر مركزها وحدة تكون اولية(شبه اولية) اذاوفقط اذا كانت اولية(شبه اولية) مركزيا.

الكلمات المفتاحية: الحلقات الأولية، الحلقات شبه الاولية، الحلقات الأولية مركزيا، الحلقات شبه الأولية مركزيا.

Introduction:

Let R be a ring .A non-empty subset S of R is said to be a multiplicative closed set in R if $a,b\in S$ implies that $ab\in S$,(Larsen and McCarthy,1971) and a multiplicative closed set S is called a multiplicative system if $0\notin S$,(Larsen and McCarthy ,1971).Let S be a multiplicative system in R such that $[S,R]=\{0\}$,where $[S,R]=\{[s,r]: s\in S, r\in R\}$. Define a relation (~) on $R\times S$ as follows :

If $(a,s),(b,t)\in R\times S$ *then* $(a,s)\sim(b,t)$ if and only if there exists $x\in S$ such that $x(at-bs)=0$ .Since $[S,R]=\{0\}$, it can be shown that (~) is an equivalence relation on $R\times S$ .Now denote the equivalence class of (a,s) in $R\times S$ by a_S, that is $a_S=\{(b,t)\in R\times S:(a,s)\sim(b,t)\}$ (this equivalence class is also denoted by $\frac{a}{s}$ (Larsen and McCarthy,1971) or by $s^{-1}a$, and then denote the set of all equivalence classes determined under this equivalence relation by R_S, that is let $R_S=\{a_s:(a,s)\in R\times S\}$ .Note that R_S is also denoted by $S^{-1}R$ (Larsen and McCarthy, 1971).

On R_S we define addition $(+)$ and multiplication $(.)$ as follows:

$a_s+b_t=(at+bs)_{st}$ and $a_s.b_t=(ab)_{st}$, for all $a_s,b_t\in R_S$.

It can be shown that these two operations are well-defined and that $(R_S,+,.)$ forms a ring which is known as the localization of R at S (Larsen and McCarthy, 1971).

Let R be a ring.Then R is called a prime ring if whenever $a,b\in R$ are such that $aRb=\{0\}$ then $a=0$ or $b=0$, (Ashraf,2005, Jung and Park, 2006), and it is called a semiprime ring if $a\in R,$ is such that $aRa=\{0\}$ then $a=0$,(Vukman, 1999, Argac, Nakajima and Albas, 2004), where $aRb=\{arb: r\in R\}$.

Before giving the main results of the paper we introduce some definitions .

Let R be a ring and S a multiplicative system in R. We say S has zero commutator if $[S,R]=\{0\}$ and we call S a bi-zero multiplicative system if:
i: $[S,R]=\{0\}$ and **ii:** $ann(s)=\{0\}$, for all $s\in S$, and we say R satisfies the bi-zero property (BZP) for multiplicative systems in R if every multiplicative system S in R which has the property $[S,R]=\{0\}$ has also the property that $ann(s)=\{0\}$, for all $s\in S$.

Example 1:
It is easy to show that every multiplicative system in Z is a bi-zero multiplicative system, that is if S is any multiplicative system in Z then $[S,R]=\{0\}$ and $ann(s)=\{0\}$, for all $s\in S$.
Now let us take the ring of all 2×2 matrices over Z, $(M_{2\times 2},+,.)$.It is known that this ring is not commutative.Take $S=\{\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\}$, it is not difficult to show that S is a multiplicative system in $M_{2\times 2}$,also we can show that $[S,M_{2\times 2}]=\{0\}$.
To show $ann(\begin{pmatrix}1&0\\0&1\end{pmatrix})=\{\begin{pmatrix}0&0\\0&0\end{pmatrix}\}=ann(\begin{pmatrix}-1&0\\0&-1\end{pmatrix})$.
If $\begin{pmatrix}x&y\\u&v\end{pmatrix}\in ann(\begin{pmatrix}1&0\\0&1\end{pmatrix})$, then $\begin{pmatrix}1&0\\0&1\end{pmatrix}.\begin{pmatrix}x&y\\u&v\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$ which implies that $\begin{pmatrix}x&y\\u&v\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$, and hence $ann(\begin{pmatrix}1&0\\0&1\end{pmatrix})=\{\begin{pmatrix}0&0\\0&0\end{pmatrix}\}$.
Similarly it can be shown that $ann(\begin{pmatrix}-1&0\\0&-1\end{pmatrix})=\{\begin{pmatrix}0&0\\0&0\end{pmatrix}\}$.That means it is also possible for non commutative rings to have multiplicative systems with above two properties.

Remark:
Let R be a ring and S a multiplicative system in R such that $[S,R]=\{0\}$.If A is an ideal of R then it is easy to show that $A_S=\{a_s : a\in A, s\in S\}$ is an ideal of R_S.
Conversly, if K is an ideal in R_S then there exists an ideal J in R such that $K=J_S$,(Jabbar,2004).It is necessary to mention that if A,B are ideals of R such that $A=B$ then $A_S=B_S$ but in general the converse is not true and we give below an example to establish this fact.

Example 2:

Consider the ring $(Z_{12}, +_{12}, \cdot_{12})$. $S = \{1,2,4,8\}$ is a multiplicative system in Z_{12}.By simple computation we can get:

$(Z_{12})_S = \{0_1, 0_2, 0_4, 0_8, 1_1, 1_2, 1_4, 1_8, 2_1, 2_2, 2_4, 2_8, \ldots, 11_1, 11_2, 11_4, 11_8\}$.

Note that

$0_1 = 0_2 = 0_4 = 0_8 = 3_1 = 3_2 = 3_4 = 3_8 = 6_1 = 6_2 = 6_4 = 6_8 = 9_1 = 9_2 = 9_4 = 9_8$,

$1_1 = 1_4 = 2_2 = 2_8 = 4_1 = 4_4 = 5_2 = 5_8 = 7_1 = 7_4 = 8_2 = 8_8 = 10_1 = 10_4 = 11_2 = 11_8$ and

$2_1 = 1_2 = 1_8 = 2_4 = 4_2 = 4_8 = 5_1 = 5_4 = 7_2 = 7_8 = 8_1 = 8_4 = 10_2 = 10_8 = 11_1 = 11_4$.

So that $(Z_{12})_S = \{0_1, 1_1, 2_1\}$ which is an ideal in $(Z_{12})_S$.

Now $I = \{0,2,4,6,8,10\}$ *and* $J = \{0,4,8\}$ are two ideals in Z_{12} so that I_S, J_S are ideals in

$(Z_{12})_S$.By the same technique as we used above we can get that

$I_S = \{0_1, 1_1, 2_1\}$ *and* $J_S = \{0_1, 1_1, 2_1\}$.So that $I_S = (Z_{12})_S = J_S$ but $I \neq J$.

and also we will show that this existence becomes unique under certain conditions as we see latter(see **Theorem 2** and **Theorem 3**).Now we mention the following two results the proof of which could be found in (Jabbar,2004).

Let R be a ring and S is a multiplicative system in R such that $[S, R] = \{0\}$.If I *and* J are ideals of R then:

1: $(IJ)_S = I_S J_S$, and **2:** $(I^n)_S = (I_S)^n$, for all $n \in Z^+$.

The Main Results:

First we prove a lemma which will play the basic role ,as we see latter, in the proof
of the main results of the paper.

Lemma 1:

Let R be a ring and S a bi-zero multiplicative system in R.If $a, b \in R$ and $s, t \in S$ then $aRb = \{0\}$ if and only if $a_s R_S b_t = \{0\}$.

Proof :

Now let $aRb = \{0\}$.Then if $r_x \in R_S$ (is any element), where $r \in R$ and $x \in S$,we have $a_s r_x b_t = (arb)_{sxt} = 0_{sxt} = 0$,(since $arb \in aRb = \{0\}$, so $arb = 0$).

Hence $a_s r_x b_t = 0$, for all $r_x \in R_S$, thus $a_s R_S b_t = \{0\}$, which proves the "only if " part.

To prove the "if"part, let $a_s R_S b_t = \{0\}$, where $a, b \in R$ and $s, t \in S$, then for any $r \in R$ we have $r_s \in R_S$, and hence $a_s r_s b_t \in a_s R_S b_t$, which gives $a_s r_s b_t = 0$ or $(arb)_{sst} = 0$, then there exists $t \in S$ (t depends on r) such that $t(arb) = 0$, thus $arb \in ann(t)$ and S bieng a bi-zero multiplicative system so

$ann(t) = 0$ and hence we get $arb = 0$, this last result is true for all $r \in R$, which implies that $aRb = \{0\}$, and this completes the proof ♦.

Remark:

If R is a ring and S is a multiplicative system in R such that $[S, R] = \{0\}$.If I , J are ideals in R such that $I = J$ then $I_S = J_S$.But in general the converse is not true (see **Example 2**). Now we give some conditions under which $I_S = J_S$ implies $I = J$.

Theorem 2:

Let R be a ring and suppose that S is a multiplicative system in R such that $[S, R] = \{0\}$.

If $I \text{ and } J$ are prime ideals of R such that $I \cap S = \phi = J \cap S$, then $I = J$ if and only if $I_S = J_S$.

Proof:

For the proof of the "only if" part see the last remark so we prove only the "if"part.

Let $I_S = J_S$ .To show $I = J$. Let $a \in I$. Since $S \neq \phi$ so take $s \in S$ .Then $a_S \in I_S$ andhence $a_S \in J_S$, and so $a_S = b_t \text{ for some } b \in J, t \in S$, so that $(a, s) \sim (b, t)$ which implies that there exists $u \in S \text{ such that } u(at - bs) = 0$, then $uat = ubs \in J$ or $uta \in J$ but $u, t \in S$ implies $ut \in S$ and $J \cap S = \phi$ thus $ut \notin J$ and J being a prime ideal so $a \in J$ .Hence $I \subseteq J$ .By the same technique we can show that $J \subseteq I$ and hence $I = J$ ♦.

Theorem 3:

Let R be a ring with identity 1, S is a multiplicative system in R such that $[S, R] = \{0\}$ and I, J are ideals in R .If every non-zero element of $CentR$ is a unit in R, then $I = J$ if and only if $I_S = J_S$.

Proof:

The "only if " part has been proved.So it remains to prove the converse part.

Let $I_S = J_S$ .If $x \in I$ then there exists $s \in S$, (since $S \neq \phi$) and then $x_S \in I_S = J_S$, and hence there exists $a_t \in J_S$, $\text{for some } a \in J \text{ and } t \in S \text{ such that } x_S = a_t$ which gives $(x, s) \sim (a, t)$,hence there exists $v \in S \text{ such that } v(xt - as) = 0$ or $vxt = vas \in J$ then $vtx \in J$ .Now $v, t \in S$ implies $vt \in S$, thus $vt \neq 0$ (since $0 \notin S$). But then since $[S, R] = \{0\}$ so $[vt, R] = \{0\}$ which means that $vt \in CentR$, hence $0 \neq vt \in CentR$ and thus by the given assumption vt is a unit in R , that is $(vt)^{-1} \in R$ and then $vtx \in J$

implies $(vt)^{-1}vtx \in J$ that is $1.x \in J$ which means that $x \in J$ and hence $I \subseteq J$ .Similarly it can be shown that $J \subseteq I$ .Hence $I = J$ ♦.

Now we introduce the following definition:

Definition:

Let R be a ring.We say that R is centrally prime (resp. centrally semiprime) if R_S is prime (resp. semiprime) for all multiplicative systems S in R which have zero commutators.

Example 3:

As we have mentioned in **Example 1,** that every multiplicative system S in Z is a bi-zero multiplicative system, that is Z satisfies the $-(BZP)$ property.Now we will show that Z_S is a prime (resp. semiprime) ring.Now let for $a,b \in Z \text{ and } s,t \in S$ we have $a_s Z_S b_t = \{0\}$, but S being a bi-zero multiplicative system so by **Lemma 1,** we get that $aZb = \{0\}$, and hence as especial case $a.1.b = 0$ or $a.b = 0$, which implies $a = 0 \text{ or } b = 0$, thus $a_s = 0_s = 0 \text{ or } b_t = 0_t = 0$, so Z_S is a prime ring , and S being arbitrary multiplicative system with zero commutator , so we get that Z is a centrally prime ring .Since every prime ring is a semiprime ring so every centrally prime ring is centrally semiprime and thus Z is also a centrally semiprime ring.

Next we apply the result of **Lemma 1,** to prove some theorems which determine the relations between prime (resp. semiprime) and centrally prime(resp.centrally semiprime) rings, in each of the following two theorems (**Theorem 4** and **Theorem 5**) a condition is given which makes prime (resp. semiprime) rings and centrally prime (resp. centrally semiprime) rings equivalent.

Theorem 4 :

Let R be a ring. If R satisfies the-(BZP) for multiplicative systems, then R is prime (resp. semiprime) if and only if R is centrally prime (resp. centrally semiprime).

Proof :

Let R be a prime ring and S be any multiplicative system in R which has zero commutator, that is $[S,R] = \{0\}$, to show R is a centrally prime ring it is enough to show that R_S is a prime ring.Since R satisfies the-(BZP) so $ann(s) = \{0\},$ for all $s \in S$, that is S is a bi-zero multiplicative system . Now let for $a_s, b_t \in R_S$ we have $a_s R_S b_t = \{0\}$, (where $a,b \in R$ and $s,t \in S$).Then by **Lemma 1,** we get $aRb = \{0\}$ and R being a prime ring we

get $a=0$ or $b=0$.If $a=0$ then $a_S=0_S=0$ and if $b=0$ then $b_t=0_t=0$ and hence R_S is a prime ring and since S is arbitrarily choosen we get that R_S is prime for each multiplicative system S in R which has zero commutator and hence R is centrally prime .

Conversely, let R be centrally prime ring, we will show that R is prime. So let for $a,b\in R$ we have $aRb=\{0\}$, if S is any multiplicative system in R with zero commutator then R_S is a prime ring and since R satisfies (BZP) for multiplicative systems so S has the property that $ann(s)=\{0\}$, for all $s\in S$, that is S is a bi-zero multiplicative system. Now since $S\neq\phi$, so there exists an $s\in S$, then $a_S,b_S\in R_S$, and since $aRb=\{0\}$ and S is a bi-zero multiplicative system so by **Lemma 1,** we get $a_SR_Sb_S=\{0\}$, but R_S being a prime ring we get $a_S=0$,or $b_S=0$. If $a_S=0$, then there exists $u\in S$ such that $ua=0$, thus $a\in ann(u)$ and since R satisfies (BZP) for multiplicative systems so $ann(u)=\{0\}$ and hence $a=0$.If $b_S=0$, by the same technique we get $b=0$. Hence R is a prime ring which completes the proof of the case when R is prime and for the case when R is semiprime the same technique is applicable to get the result ♦.

Remark:

In the **Example 3,** we have proved directly that Z is a centrally prime as well as a centrally semiprime ring, here we can use **Theorem 4,** to show this fact as follows:

It is known that a non-zero ring which has no zero divisors are prime as well as semiprime and since Z has no zero divisors so it is prime and hence semiprime,on the other hand Z satisfies the (BZP) for multiplicative systems as we have mention in **Example 3,** thus by applying **Theorem 4,** Z becomes centrally prime and hence centrally semiprime .

Theorem 5:

Let R be a ring.If R has the identity 1 and every nonzero element of $CentR$ is a unit in R then R is a prime (resp. semiprime) ring if and only if it is centrally prime (resp. centrally semiprime).

Proof:

We will show that R satisfies the-(BZP) for multiplicative systems in R,so let S be any multiplicative system in R with zero commutator.If $s\in S$ is any element then since $[S,R]=\{0\}$ so $[s,r]=0$, for all $r\in R$, that is $sr-rs=0$, for all $r\in R$, and thus $sr=rs$, for all $r\in R$ which means that $s\in CentR$, and then $0\notin S$ *and* $s\in S$ implies $s\neq 0$, hence $0\neq s\in CentR$ which means that s is a unit , the next step is to show that $ann(s)=\{0\}$, so let

$x \in ann(s)$ then $sx = 0$ and s being a unit , $s^{-1} \in R$ thus $sx = 0$ implies that $s^{-1}sx = s^{-1}0 = 0$, hence $x = 0$ which means $ann(s) = \{0\}$, and s being arbitrary in S so we get that $ann(s) = \{0\}$, for all $s \in S$, and hence every multiplicative system S in R with zero commutator satisfies also the property $ann(s) = \{0\}$, for all $s \in S$, and thus R satisfies (BZP) for multiplicative systems.Hence by **Theorem 4,** R is prime (resp. semiprime) if and only if it is centrally prime (resp.centrally semiprime) which completes the proof ♦.

Now, in each of the following two theorems we give a condition which makes prime (resp. semiprime) rings centrally prime (resp. centrally semiprime),we see below that nonzero prime (resp. semiprime) rings which have no zero divisors are centrally prime (resp. centrally semiprime), which means , in some sense , that centrally prime (resp. centrally semiprime) rings are generalizations of those non-zero rings which have no proper zero divisors.

Theorem 6 :

If R is a non-zero prime (resp. semiprime) ring which has no proper zero divisors then it is centrally Prime (resp. centrally semiprime).

Proof:

Suppose R has no proper zero divisors.We will show that R is centrally prime , so let S be any multiplicative system in R with zero commutator , to show that R_S is a prime ring, let for $a_s, b_t \in R_S$ we have $a_s R_S b_t = \{0\}$, where $a, b \in R$ and $s, t \in S$. Since $R \neq \{0\}$, so there exists $0 \neq r \in R$. Then $r_s \in R_S$ and hence $a_s r_s b_t \in a_s R_S b_t$ which gives $a_s r_s b_t = 0$ or $(arb)_{sst} = 0$, and hence we get that there exists $t \in S$ such that $t(arb) = 0$ or $tarb = 0$, but R has no proper zero divisors so $t = 0$ or $a = 0$ or $r = 0$ or $b = 0$.But $0 \notin S$ *and* $t \in S$ implies that $t \neq 0$.Also $r \neq 0$ (since r is choosen non-zero in R) thus we get $a = 0$ or $b = 0$. If $a = 0$ then $a_S = 0_S = 0$ and if $b = 0$ then $b_t = 0_t = 0$.

Hence R_S is a prime ring and S being arbitrary multiplicative system in R with zero commutator we get R_S which is a prime ring for all multiplicative systems S in R with zero commutators and hence R is a centrally prime ring.The proof of semiprimeness case is exactly as the proof of primeness case but we just take $a_s R_S a_s = \{0\}$ instead of $a_s R_S b_t = \{0\}$ and repeating the same outlines of the above proof ♦.

Next we give another condition under which prime (resp. semiprime) rings are centrally

prime (resp. centrally semiprime) and that condition provides R to be a finite ring and this can regarded as a corollary to **Theorem 6**.

Corollary 7 :

A finite prime (resp.semiprime) ring R is centrally prime (resp. centrally semiprime).

Proof:

We will show only the case when R is a prime ring and the case when R is semiprime can be done by the same technique. So let $R=\{r_1, r_2,...,r_n\}$ and S be any multiplicative system in R with zero commutator, we must show that R_S is a prime ring.Now let $a_s, b_t \in R_S$ are such that $a_s R_S b_t = \{0\}$, where $a, b \in R$ and $s, t \in S$.

For each i $(1 \le i \le n)$ we have $(r_i)_s \in R_S$ and thus $a_s (r_i)_s b_t = 0,$ for all i, or $(ar_i b)_{sst} = 0,$ for all i,and thus we get that for each i $(1 \le i \le n)$**,** there exists $t_i \in S$ such that $t_i (ar_i b) = 0$.Now let $x = t_1 t_2 .. t_n$. Since for all $i,\ t_i \in S$ so that $x = t_1 t_2 .. t_n \in S$. But since $[S, R] = \{0\}$ and for all i, we have $t_i \in S$**,** so $t_i t_j = t_j t_i$**,** for all i, j.Hence for each i we get

$x(ar_i b) = t_1 t_2 .. t_n (ar_i b) = t_1 t_2 .. t_{i-1} t_{i+1} .. t_n t_i (ar_i b) = t_1 t_2 .. t_{i-1} t_{i+1} .. t_n .0 = 0$,

which gives that $(xa) r_i b = 0,$ for all i, that is $(xa) r_i b = 0,$ for all $r_i \in R$ and this means $(xa) R b = \{0\}$, but R being a prime ring we get $xa = 0$ or $b = 0$.

If $xa = 0$ then $a_S = x_x a_S = (xa)_{xs} = 0_{xs} = 0$, and if $b = 0$ then $b_t = 0_t = 0$.

Hence R_S is a prime ring which proves that R is centrally prime ♦.

In fact, **Corollary 7,** tells us that centrally prime (resp.centrally semiprime) rings are, in some sense, generalizations of finite prime (resp. semiprime) rings.

REFERENCES

[1] Argac N.,Nakajima A. and Albas E. ;(2004)“ On Orthogonal Generalized Derivations of Semiprime Rings” Turk J Math ,28,185-194.

[2] Ashraf M. ;(2005) “On Left $(\vartheta-\phi)$ -Derivations of Prime Rings” Archivum Mathematicum (BRNO) Tomus 41,157-166.

[3] Jabbar A.K. ;(2004)”Almost Noetherian Domains which are Almost Dedekind” KAJ, 3(1) Part A , 33-39.

[4] Larsen M.D. and McCarthy P.J. ;(1971) ”Multiplicative Theory of Ideals”Academic Press New York and London .

[5] Jung Y.S. and Park K.H. ;(2006)” On generalized $(\alpha-\beta)-$derivations and commutativity in prime rings” Bull. Korean Math. Soc. 43 , No.1, 101-106.

[6] Vukman J. ;(1999)”An identity related to centralizers in semiprime rings” Comment. Math. Univ. Carolinae 40, No.3, 447-456.