On MP–rings and DS–rings

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Abstract:

The research aims to study two kinds of rings,MP-rings and DS-rings . The researcher gave some binding relation with other modules and rings . The researcher put the hypothesis that condition.

على الحلقات من النمط-MP والحلقات من النمط-DS

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ملخص البحث :

يهدف البحث إلى دراسة نوعين من الحلقات هي الحلقات اليسرى من النمط−MP ،والحلقات اليسرى من النمط−DS .وأعطينا بعض العلاقات التي تربط بين هذه الحلقات وحلقات أخرى ،وأعطينا شرط(*) في أثبات علاقة الحلقات اليسرى من النمط−DS مع حلقات أخرى .

1. Introduction :

To study a left MP-rings and a left DS-rings[3] requires our knowledge of other definitions as :

1. A right R-module M is said to be P-injective if and only if ,for each principal right ideal I of R ,and every right R- homomorphism $f:I\rightarrow M$, there exists y in M such that $f(x)=yx$ for all x in I,[2].
2. An R-module M is called simple when its only sub modules are 0 and M, [5].

3. A right annihilator of a non-zero element a in a ring R is defined by r(a)={b∈R: ab=0},a left annihilator l(a) is similarly defined,[5].
4. An R-module M is called faithful if and only if Ann(M)=0,[5].

Many scientists studied rings such Nicholson,Yousif and Watter, Nicholson proved that [3](Every MP-ring is DS-ring) ,if R is P-injective or R is commutative. So (Every MP-ring is A left faith R-module).

2- MP–Rings

Following [3]

A ring R is called a left MP-ring if every minimal left ideal of R is a P-injective module. And a left R–module M is called an MP-module if every simple sub module is a P-injective module .

Theorem (2-1)[3]

The following conditions are equivalent for a ring R :-

1. R is a left MP-ring.
2. R is has a faithful left MP–module.
3. If L is a maximal left ideal of R then either r(L) =0 or R/L is P–injective module.
4. Every simple left R-module K either is P-injective or satisfies hom(K,R)=0.

Lemma (2-2)

Let R be a left MP-ring , if for each minimal left ideal L of R ,and every $0 \neq a \in R$ Then $r(l(a)) = aL$.

Proof :

Let L be a minimal left ideal of R and 0 ≠a Є R since $_RL$ is P-injective, then r(l(a)) = aL.

Theorem (2-3)

Let R be a left MP-ring, such that for each minimal left ideal L of R, and every 0 ≠ a Є R. If f: Ra → L is any R-linear map, then f(a) Є aL.

Proof :

Let L be a minimal left ideal of R and 0 ≠ a Є R

Then rl(a) = aL

If f: Ra→ L is any R-linear map, Then:

l(a) f(a) = f(l(a)a) = f(0) = 0

So f(a) Є rl(a) = aL

Then f(a) Є aL

Theorm (2-4)

Let R be a left MP-ring, such that for each minimal left ideal L of R and every 0 ≠ a Є R, if l(a) ⊆ l(k), where 0 ≠ k∈ L, then 0 ≠ kL ⊆ aL

Proof :

Let L be a minimal left ideal of R and 0 ≠ a ∈ R, if l(a) ⊆ l(k), where kЄL, then kL = rl(k) ⊆ rl(a) = aL, L=Re, e^2=e∈R,since k = ke∈ kL, kL≠0 then 0 ≠ kL ⊆ aL

Theorem (2-5)

The following conditions are equivalent

1. R is a left MP-ring.

2. For each minimal ideal L of R and every $0\neq a\in R$, $r(Rb\cap l(a)) = r(b)+ aL$

Proof :

$1 \longrightarrow 2$

Let L be a minimal left ideal of R and $0\neq a$, $b\in R$.

We can suppose $a\in r(b)+ aL$, we know $a\in r(b)\subset r(Rb)$, so $a\in aL=r(l(a))$

Therefore $a\in r(Rb)\cap r(l(a))\subseteq r(Rb\cap l(a)$

$\therefore$ $r(Rb\cap l(a)) \supseteq r(b) + aL$ ……..1

Now suppose $x\in r (Rb\cap l(a))$ ………2

Then $l(ba)\subseteq l(bx)$. If $bx=0$ then $x\in r(b) + aL$

If $bx\neq 0$ then by theorem (2-4) $0\neq bxL\subseteq baL$

So $L=Re$,from the same theorem, where $e^2=e$.

Hence $bx=bxe\in bxL\subseteq baL$, $bx=bay$, where $y\in L$.

Then $b(x-ay)=0$ and $x-ay\in r(b)$.

Hence $x\in r(b) + aL$ ………3

From(2),(3) we get $r(Rb\cap l(a))\subseteq r(b) + aL$ ……..4,

from (1),(4)

Then $r(Rb\cap l(a)) = r(b) + aL$.

$2 \longrightarrow 1$

If for every minimal L left ideal L of R and $0\neq a$, $b\in R$

We have $r(Rb\cap l(a)) = r(b) +aL$.

Then we let $b=1$ and then $rl(a) = aL$.

Hence R is a left MP-ring by theorem (2-2).

3- DS-ring:

Following [3]

A ring R is called a left DS-ring if every minimal left ideal of R is a direct summand.

Definition (3-1) [4]

A ring R is called a right (left) minijective ring if and only if for any minimal right (left) ideal E of R. every R-homomorphism of E into R extends to one of right (left) R into R.

Following [3] A left R-module M is called a DS-module if every simple sub module is a minijective.

Definition (3-2)

A right R has condition (*) if $K \cong Re$ are simple, $e^2=e$, then $K=Rg$ for some $g^2=g$.

Obviously a left DS-ring and a left minijective ring have condition (*).

Lemma (3-3)[3]

The following condition are equivalent:

1. R is a left Ds-ring.
2. Soc(R) is a minijective module.
3. R has a faithful left DS-module.

Theorem (3-4)

If L is a maximal left ideal of R and either $r(L)=0$ or R/L is a minijective module then R is a left DS-ring.

Proof :

Let Rk be a minimal left of R.

If $k^2 \neq 0$, then $Rk = Re$, e being an idempotent otherwise $k \in l(k)$, and then $R/l(k)$ is a minijective let $f: Rk \rightarrow R/l(k)$, by $f(rk) = r+l(k)$, then there exists a$d \in R$ such that $1-kd \in l(k)$

Henece $k=kdk$.

Let $g=dk$, then g is an idempotent ,and $Rk=Rkdk = Rkg \subseteq Rg = Rdk \subseteq Rk$.

Hence Rk = Rg.

Therefore R is a left DS-ring, Definition (3-2)

Theorem (3-5)

A ring R is a left DS-ring if and only if $J(R)\cap soc(R)=0$.

Proof :

Let R be a left DS-ring, If $J(R)\cap Soc(R)\neq 0$ then there exists a minimal left ideal M of R with $M\subseteq J(R)$.

But M=re for some $0\neq e^2 = e\in R$

So $e\in J(R)$, a contradiction

Therefore $J(R)\cap soc(R)=0$

Conversely:

If M is a minimal left ideal of r, then $J(R)\cap Soc(R) =0$, implies $M^2\neq 0$

So M=Re, where $e^2=e\in R$. Thus R is a left DS-ring.

Definition (3-6) [2]

Let R be a ring and x be an element in R, then x is said to be left singular if and only if L(x) is essential ideal in R. The set of all left singular elements in R is denotef by Z(R).

Z(R) is an ideal in R which is the left singular ideal of R.

Definition (3-7)

A ring R is said to be SSM-ring if and only if every singular simple left R-module is minijective.

Theorem (3-8)

Let R is a SSM-ring and has condition (*), then R is a left DS-ring.

Proof :

Let Rk be a minimal left ideal of R

l(k) be a maximal left ideal of R.

If l(k) is not essential. Then l(k) is a direct summand of R.

Hence Rk≅R/l(k) is projective, Rk ≅Re, where e^2=e.

Then Rk=Rg ,since R has condition (*) where g=g^2 .

If l(k) is essential, then Rk is singular simple so is minjective and we easily show that k=kdk [by proof Lemma (3-3)] let e=dk

Then Rk=Re and e^2=e

Then R is a left DS-ring.

Theorem (3-9)

A sub direct product of a left DS-ring is a gain a left DS-ring.

Proof :

Let R /A_i be a left DS-ring for each i∈I where $\cap_{i\in I}$ A_i=0 If M is a minimal left ideal of R, Then M⊄Ai for some I , So (M+A_i)/A_i is a minimal left ideal of R/A_i. It follows from Theorem (3-3) that M^2⊄A_i So M^2=M.

Hence M=Re where e^2=e and then R is a left DS-ring.

Theorem (3-10)

A ring R is a left DS-ring if and only if for each minimal left ideal K is K ⊄ r(K).

Proof :

Suppose that R is a left DS-ring ,

Let K be a minimal left ideal of R. Then K=Re, where e is an idempotent. Hence $K^2 \neq 0$, and $K \not\subset r(K)$.

Conversely,

If for each minimal left ideal K of R, $K \not\subset r(K)$,

Then $K^2 \neq 0$ and $K^2=K$, so K=Re, $e^2=e$, R is a left DS-ring**.**

Definition (3-11) [1]

A left R-module M is said to be flat if for any monomorphism N $\rightarrow$Q of right R-module N, Q, the induced homomorphism $N \otimes M \rightarrow Q \otimes M$ is also homomorphism.

Theorem (3-12)

If J(R) ∩ Soc(R) is a flat left R-module, then R is a left DS-ring.

Proof :

Let $\{M_i \backslash\, i \in \Omega\}$ be a set of representative of non-isomorphic class of simple right R-module and $U=\Sigma_{i \in \Omega} \otimes M_i$.

Then we have an exact sequence:

L: $0 \rightarrow U \rightarrow E(U) \rightarrow E(U)/U \rightarrow 0$.

Where E(U) is the injective hull of U. Since $_R(J(R) \cap Soc\ (R))$ is flat, $0=U(J(R)) \cap Soc\ (_RR))= E(U)(J(R) \cap Soc\ (_RR)) \cap U$.

Hence $E(U)(J(R) \cap Soc\ (_RR))=0$

As U is essential in E(U) , Since E(U) is an injective [see(2)] co generator it is faithful so $J(R) \cap Soc(_RR)=0$

By theorem (3-5), R is a left DS-ring.

References

1. Ming, R.Y.(1997);" A note On Yj – injective ",Demon Math .Vol xxx No. 3,551-556 .
2. Ming, R.Y.(1973);"On Simple P-injective Modules",Math.Japonica. 19,173-176.
3. Nicholson,W.K. Watters,J.F. (1988);" Rings With Projective Socle ",Proc. Amer ,Math.Soc. 102,443-450 .
4. Nicholson,W.K., Yousif ,M. (1997);" Minijective Rings ",Algebra, J. 187,548-578 .
5. Shoker,N.H. (1974);"On V-rings",M.Sc.Thesis Baghdad University.